ELEMENTS OF EUCLID.

PART I.

THE

ELEMENTS OF EUCLID,

WITH

MANY ADDITIONAL PROPOSITIONS,

AND

EXPLANATORY NOTES;

TO WHICH IS PREFIXED AN

INTRODUCTORY ESSAY ON LOGIC.

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PART I.

CONTAINING THE FIRST THREE BOOKS.

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PREFACE.

In presenting to the public a new elementary work on the Principles of Geometry, it can hardly be necessary to defend the having made Euclid's Elements the basis of the work; for while it cannot be denied that many grave faults exist even in the best translations, and that, owing to the advances made in mathematical science since Euclid's day, the demonstrations of many important theorems are wanting in the Elements; it must, on the other hand, be acknowledged that, notwithstanding the numerous attempts which have been made by our best modern geometers to supersede it, the Elements has ever held the chief place in all our universities and colleges.

In the present edition the text of Dr. Simson has been principally followed, but occasionally preference has been given to that of Elrington; the whole has, however, been entirely rewritten, and it is hoped that, in the attempt to render it less verbose, it will not be found that the chain of proof has been in any case weakened. Considerable pains have been taken to distinguish the various parts of the propositions by the adoption of differences in the type; and the references have been grouped in tables under the diagrams, in order to show at sight upon which preceding theorems the truth of each depends.

In the explanatory notes which have been appended it will be found that many additional propositions have been added, and that in several instances other demonstrations have been given.

In the second book it has been endeavoured to point out the relative connection of Geometry and Algebra, and to illustrate by the former the theory of quadratic equations.

In order to remove one of the most practical objections which have been urged against the Elements, namely, its want of methodical arrangement, a classified index has been appended, by

means of which the theorems relating to any particular subject may be immediately found.

In conclusion it must not be omitted to mention the works which have been principally consulted, and to which the present edition must be considered as mainly indebted for any advantages which it may possess. These have been the various editions of the Elements by Simson, Elrington, Tacquet, Barrow, De Chales, Lardner, Potts, Byrne, Playfair, and Thomson, Leslie's Elements of Geometry, Wright's Self-examinations in Euclid, Cresswell's Treatise on Geometry, Bonnycastle's Elements of Geometry, the volume on Geometry in the Library of Useful Knowledge, and a most valuable paper by Professor De Morgan in the Companion to the British Almanac, entitled "Short Supplementary Remarks on the First Six Books of Euclid's Elements."

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6, *Duke Street, Adelphi,*
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INTRODUCTION.

The object of Geometry is to investigate and deduce by strict Logic those relations and properties of space and figure which they possess, irrespective of any properties of a physical nature. The whole science of Geometry is based upon certain simple and self-evident truths, from which, by a continuous chain of reasoning, conducted strictly in accordance with the rules of logic, the most important and complicated relations of space and figure are deduced and demonstrated. It is the only science in which hypotheses and theories are unknown, to which experiment and experience have rendered no aid, and whose conclusions are certain and immutable. However much the rules of logic may assist in obtaining true conclusions in the investigations of experimental science, it is only in those of Geometry that its laws are never departed from.

It is therefore evident that the student of Geometry should be perfectly acquainted with formal logic; and we shall give a brief outline of that science before proceeding to the more immediate object of this work. Much discussion has taken place amongst writers on logic as to the scope of the science; some contending that logic includes in its object all the operations of the human understanding necessary to the pursuit of truth; while others would limit it merely to a collection of general rules, by means of which true conclusions may infallibly be derived from true premises. For our present purpose it will suffice to treat the subject in its more limited sense, without entering into the consideration of questions which the first-named writers consider as belonging rather to metaphysics than logic.

The object of reasoning is to extend our knowledge; to enable us, from certain known facts, to derive others of a more general nature; from premises whose truth is evident and acknowledged, to demonstrate the truth of conclusions not in themselves self-evident, and frequently such as, without such proof, would have been regarded as false. In every process of reasoning there are two distinct points to be attended to, namely—

1st. That the propositions employed as premises are not ambiguous, are correctly understood, and are true.

2nd. That the steps by which a conclusion is drawn from those premises are true.

The subject therefore ranges itself properly under two heads; namely, first, an examination into the nature and meaning of propositions, or those premises upon which our reasoning is to be founded; and secondly, an investigation into the mode or form of reasoning to be adopted, that the conclusions drawn may be as true as the premises.

A *proposition* may be defined to be "An assertion, affirming or denying something;" and, as Mills has justly remarked, "whatever can be an object of belief, or even of disbelief, must, when put into words, assume the form of a *proposition*;" so that we can never make an assertion, or even hazard a conjecture, without expressing one or more propositions. Now it will be found that the simplest form in which a proposition can exist is the bare statement of the possession of some property, quality, or circumstance, by something. Two objects must be concerned; the something which is the subject of discourse, and the something which is asserted in relation to it; and the proposition is nothing more than the statement of their relative connection. Thus every proposition consists of three parts; namely, 1st, the something of which the assertion is made, termed the *subject;* 2nd, the sign of affirmation or denial, called the *copula;* and 3rd, the property, quality, or circumstance asserted, named the *predicate.* For example, the assertion that "the sun is round" is a proposition of which "the sun" is the *subject*, the verb "is" the *copula*, and "round" the *predicate.* Again the exclamation "I think so" is a perfect proposition, being equivalent to "such is what I think," in which the word "such" is the *subject*, and the phrase or sentence "what I think" the *predicate.* And here it should be observed that the subject and predicate of a proposition may be either simple terms, or names given to objects or their attributes, or they may be complex sentences, themselves containing other propositions. In either case, however, it is essential that the meaning of each should be definite and precise, and perfectly understood, to insure which it is essential that every name or term employed should have but one meaning attached to it, and that that meaning should be perfectly known and undorstood.

Hobbes has rightly defined a *name* to be "a word taken at pleasure to serve for a mark, which may raise in our mind a thought like to some thought which we had before, and which being pronounced to others, may be to them a sign of what thought the speaker had before in his mind." This definition not only states very precisely what a name is, but shows its use and object, which is simply to suggest to the minds of the speaker and hearer the idea to which it had been attached by the common consent of both. It is therefore evident, that if any word has more than one meaning attached to it, or a meaning unknown to the person who hears it, no certain information can be conveyed to his mind by that word, and it will fail to raise up any certain definite thought or idea; whereas, on the other hand, if that

word has already been identified in his mind with some one single definite object, the mention of the word will suffice to recall that object, and with certainty inform him of what it is that we speak.

In precise language, therefore, to avoid ambiguity, only one term should be used to express the same idea, and only one idea should be comprehended under the same term; and further, the precise idea to which the term was attached should be distinctly and positively defined. Now, the definition of a term is the explanation of the meaning of that term, expressed by other terms which are already understood, and not synonymous with the term to be defined.

There are two kinds of ideas, namely, *simple* and *complex;* the first are those which exist in the mind single and independent of every other idea, and can only be expressed by naming the term employed to denote them; such are some of the terms employed in Geometry, as *point, space,* &c.: the second are such as result from the combination or comparison of two or more simple ideas, and can be expressed in either of two ways, namely, firstly, by naming the term expressing them, or, secondly, by expressing, in the proper terms, the manner in which the complex idea is formed from other simple ideas; that is, in other words, by defining the complex idea.

The simplest form in which a definition can be expressed is, a statement of the *class* to which the term to be defined belongs, and of that property which distinguishes it from every other of that class; the first is called the *genus*, and the second the *differentia* of the definition. For example, the definition of a parallelogram, given at page 5, is as follows:—"A PARALLELOGRAM is a quadrilateral figure, *whose opposite sides are parallel;*" here the *genus* or class to which a parallelogram belongs is that of quadrilateral figures, and the *differentia* or peculiar property which distinguishes a parallelogram from every other quadrilateral figure is, that its opposite sides are parallel. In the definitions throughout this work, the thing to be defined, the genus, and the differentia, are all distinguished by a difference of type; thus the thing to be defined is printed in small capitals, the *genus* in Roman, and the *differentia* in Italics.

Now, the *genus* and *differentia* must each involve a distinct idea, and since every simple idea is entirely independent of every other idea, it is evidently impossible to define it logically; hence (as is stated at page 1) it is that no logical definition can be given of such things as a *point, line,* or *surface.*

The thing to be defined can only belong to one class, but it may be distinguished from every other in that class by more than one property peculiar to itself. Hence, in the definition of a term, there can be but one genus, but there may be several differentiæ. It would not, however, be necessary to state more than one differentia in order to give a correct and sufficient logical defini-

tion. And in the mathematics, especially, only one differentia should be stated in the definition, and that one should be the easiest to be expressed and understood; its other differentiæ (or peculiar properties) should be afterwards shown, by logical deductions founded on the previous definition and some other admitted truths, to belong to the thing defined.

When, in the definition of a term, more than one peculiar property or differentia is stated, it ceases then to be a logical definition, and becomes a *description.* The object of a description is to convey to the mind a complete notion of everything included in the complex idea described, so that the mind may perceive or take in, at one view, the whole complex idea in its full extent and generality; that is, may perceive, at one time, every other idea which it is meant to include. While, on the other hand, the object of a definition is rather to limit the mental perception, and fix it upon some one peculiar property or differentia, and to none other; to enable the mind with certainty to separate that idea from every other, and view it distinctly with reference to any one (but only one) of its distinguishing features.

By thus attending to the precise meaning of the terms employed in the subject and predicate of our propositions, we are enabled to fulfil the first requirement which we laid down as necessary to the attainment of a true conclusion, namely, "that the propositions employed as premises are not ambiguous, are correctly understood, and are true."

We have said that every assertion involves a proposition, but it is seldom that the propositions so involved are explicitly stated in such a form as to enable us to distinguish at once the subject, predicate, and copula. Whatever the form, however, of the original proposition, it may always be so expressed that the subject and predicate shall be separated by the copula, which latter may always be reduced to the present tense of the substantive-verb TO BE, namely, either "is" or "is not," "are" or "are not." Thus if the original proposition were, "I have learned geometry," it might be reduced to the form "Geometry IS *a science which I have learned;*" or if it were "Proficiency in mathematics can only be attained by long study," it might be reduced to "Proficiency in mathematics IS *only to be attained by long study.*" In these examples the subject is distinguished by being printed in Roman, the copula by being in small capitals, and the predicate by being in Italics, and this mode of distinguishing the several parts of every proposition will be used in the following pages.

As regards the truth or falsity of a proposition, logic has no more to do than to see that the terms employed in its subject and predicate are distinctly defined in meaning, so that the assertion made may be clearly and correctly understood; whether the assertion so made be true or false, it is not the province of logic to inquire, but of that particular branch of science or knowledge to which the subject of the proposition relates or belongs. In the propositions

which we shall employ in the following pages, to explain their use in logical reasoning, we shall therefore substitute letters as symbols, to represent the subject and predicate. Thus, if we put the letter W to represent the science of geometry and distinguish all the sciences which I have learned by the letter Y, the above proposition, "Geometry IS *a science which I have learned*," may be expressed by "W IS *Y*;" and if X be made to stand for "Proficiency in mathematics," and Z for "anything which is only to be attained by long study," the second proposition may be more briefly expressed by "X IS *Z*."

Propositions regarded as sentences may be divided according to their grammatical structure into *categorical* and *hypothetical;* a categorical proposition makes a simple assertion, as "Every square IS *a parallelogram;*" an hypothetical proposition may be either *conditional*, that is, when the assertion is made under a condition, as, "If a triangle is equilateral, it is equiangular," or *disjunctive;* that is, when the assertion involves an alternative, as, "A right-angled triangle must be either isosceles or scalene." It should be observed, that most of the theorems in Euclid are in the form of *conditional hypothetical propositions.*

Again, as it is the essential *quality* of an assertion to affirm or deny, propositions are divided according to their *quality* into *affirmative* and *negative:* thus, "The opposite sides of a square ARE *equal to each other*," is an *affirmative* proposition, and "One side of a triangle IS NOT *equal to the sum of the other two*," is a *negative* proposition.

The third division of propositions is according to their *quantity* into *universal* and *particular.* A proposition is said to be *universal* when the predicate refers to the *whole* of the subject, as, "Every equiangular triangle IS *equilateral;*" here the quality of being equilateral is predicated of the whole of the class of equiangular triangles; if, however, the predicate refers only to a *portion* of the subject, then the proposition is said to be *particular*, as, "Some triangles ARE *isosceles;*" here the property of being isosceles is only predicated of certain particular triangles included in the term some, and there may be other triangles of which that property could not be predicated.

The following table exhibits at one view the threefold division explained above, and affords an example of each kind of proposition:—

PROPOSITIONS,

Considered as *sentences**, may be divided into:—

Categorical	. . . . . .	*as* X IS *Y*.
Hypothetical	Conditional	*as* If X is Y it is Z.
	Disjunctive	*as* X is either Y or Z.

* This is said to be a division of propositions according to their substance.

According to *quality*, may be divided into :—

{Affirmative . . . . *as* X IS *Y*.
{Negative . . . . *as* X IS NOT *Y*.

And according to *quantity*, may be divided into :—

{Universal . . . . *as* Every X IS *Y*.
{Particular . . . . *as* Some Xs ARE *Ys*.

As every proposition must be either affirmative or negative, and also either universal or particular, with the same subject and predicate four different propositions may be framed, which, for convenience, are distinguished by the four vowels, A, E, I, and O; the following table enumerates them, and gives an example of each :—

A	Universal Affirmative . .	Every X IS *Y*.
E	Universal Negativo . . .	No X IS *Y*.
I	Particular Affirmative . .	Some Xs ARE *Ys*.
O	Particular Negative . . .	Some Xs ARE NOT *Ys*.

When the *terms* of a proposition, that is, the subject or predicate, include or relate to everything which can be referred to by such terms, they are said to be *distributed*, and if the contrary, *non-distributed;* thus in the proposition, "Every man IS *mortal*," the subject, "every man," is said to be distributed, because the quality of mortality is asserted to belong to *everything* to which the name of man can be applied; but the predicate, "mortal," is not distributed, because the proposition does not make any assertion with regard to *everything* that can be called mortal. The distribution of the *subject* depends upon the *quantity* of the proposition; in a *universal* proposition the *subject* is always distributed, but never in a particular one; thus in the universal affirmative proposition, "Every X IS *Y*," or the universal negative, "No X IS *Y*," the subject is distributed, because the assertion is in each case made of everything to which the symbol X can be applied; whereas in the particular affirmative proposition, "Some Xs ARE *Ys*," or the particular negative, "Some Xs ARE NOT *Ys*," the subject is not distributed, because the assertions are only made of a portion,—some,—of the things to which the symbol X refers. The distribution of the *predicate* depends upon the *quality* of the proposition, the *predicate* being always distributed in a *negative* proposition, and never in an affirmative one; thus in the universal negative proposition, "No X IS *Y*," or the particular negative, "Some Xs ARE NOT *Ys*," the predicate is distributed, because the assertion made is of everything which can be represonted by the symbol Y; and in the universal affirmative proposition, "Every X IS *Y*," or the particular affirmative, "Some Xs ARE *Ys*," the predicato is not distributed, because no assertion is made of the whole of the class which the symbol Y represents.

When two propositions which have the same subject and predicato differ in quality or quantity, or both, they are said to be

opposed to each other. If they differ in both quality and quantity they are called *contradictories*, as

A Every X IS *Y*.
O Some Xs ARE NOT *Ys*.

Or E No X IS *Y*.
I Some Xs ARE *Ys*.

If two *universal* propositions differ in quality they are called *Contraries*, as

A Every X IS *Y*.
E No X is *Y*.

If two *particular* propositions differ in quality they are called *Subcontraries*, as

I Some Xs ARE *Ys*.
O Some Xs ARE NOT *Ys*.

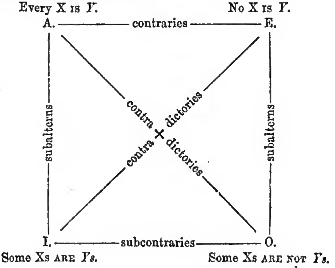
If two proportions agree in quality but differ in quantity, they are called *subalterns*, but they are not actually opposed to each other, as

A Every X IS *Y*.
I Some Xs ARE *Ys*.

Or E No X IS *Y*.
O Some Xs ARE NOT *Ys*.

As regards opposed propositions, it may be observed that
In *Contradictories*, one must be true and the other false.
In *Contraries*, both may be false, but they cannot both be true.
And in *Subcontraries*, both may be true, but they cannot both be false.

The following scheme will exhibit at one view the various kinds of opposition alluded to above :—



The *conversion* of a proposition is the changing the relative positions of the subject and predicate, so that the predicate may become the subject, and the subject the predicate of the new proposition: and propositions thus related, namely, having the subject of one the predicate of the other, and *vice versâ*, are called *converse* propositions, thus, "Every equiangular triangle IS *equilateral*," and "Every equilateral triangle IS *equiangular*," are *converse* propositions. When the truth of the converse is implied by that of the original proposition, the conversion is said to be *illative*, but this is only the case under certain circumstances, namely, when the subject and predicate are precisely similar in quantity, or, in other words, are equally distributed, being either both universal or both particular. The following table shows the distribution of the subject and predicate in each kind of proposition:—

Quality and Quantity of Proposition.	Quantity of Subject.	Quantity of Predicate.
A. Universal Affirmative.	Distributed. .	Not distributed.
E. Universal Negative .	Distributed. .	Distributed.
I. Particular Affirmative.	Not distributed	Not distributed.
O. Particular Negative .	Not distributed	Distributed.

From this table it will be seen that in the universal negative proposition, and in the particular affirmative, both the subject and predicate are similar in quantity, in the former both being universal or distributed, and in the latter both particular or not distributed; both these forms of proposition may therefore be illatively converted, that is to say, if the truth of a universal negative or of a particular affirmative proposition be admitted, the truth of its converse is implied and cannot be denied. In regard, however, to the other two sorts of propositions, namely, the universal affirmative and the particular negative, since the subject and predicate are not equally distributed, they are not *illatively* convertible, that is to say, the truth of the converse is not implied by the truth of the original proposition, and therefore cannot be inferred from it, but must be the subject of separate proof. Thus, in the Elements, the truth of the converse of the universal affirmative proposition, "Every equilateral triangle IS *equiangular*," cannot be inferred, but is made the subject of separate proof in the next proposition.

In the conversion of a universal negative or a particular affirmative proposition, both the quantity and quality of the proposition remain unaltered; thus, the converse of E gives E, and the converse of I gives I; and this kind of conversion is called *simple*

conversion. Although a universal affirmative proposition cannot be illatively converted by simple conversion, it may be by a process which is termed *conversion per accidens*, namely, by diminishing its quantity (its quality remaining unaltered) so that its converse may be only a particular affirmative proposition; for instance, the universal affirmative proposition, "Every X IS *Y*," may be illatively converted, *per accidens*, into the particular affirmative, "Some Ys ARE *Xs*," the truth of this latter being positively implied by that of the former proposition.

Having thus briefly examined the nature of propositions, we must pass on to the second division of the subject, namely, the mode or form of reasoning to be adopted, in order that the truth of the conclusion may be as certain as that of the propositions employed.

Now there is but one regular form in which an argument can be logically stated, and that form is called a *syllogism*. A syllogism consists of three propositions so related that the truth of the last (termed the *conclusion*) is manifestly established, by admitting the truth of the two first (termed the *premises*).

In the proposition which is the *conclusion* of the syllogism, the predicate is called the *major term* and the subject the *minor term*, and these two are termed the *extremes*. In one of the other propositions or premises the major term is compared with some other term called the *middle term*, and this proposition is called the *major premiss;* in the other proposition the minor term is compared with the same middle term, and this is called the *minor premiss*. Now in both the premises the middle term may be either the subject or predicate, and hence arises what is termed the difference in the *figure* of the syllogism.

In the first figure the middle term is the subject of the major premiss and the predicate of the minor, for example*:

[Major Premiss] Every (plane figure) IS *bounded by lines.*
[Minor Premiss] Every triangle IS *a (plane figure)*,
therefore
[Conclusion] Every triangle IS *bounded by lines.*

In the second figure the middle term is the predicate of both the premises, as in the following example:—

[Major Premiss] No circle IS *(bounded by straight lines)*.
[Minor Premiss] Every square IS *(bounded by straight lines)*,
therefore
[Conclusion] No square IS *a circle.*

In the third figure the middle term is the subject in both premises, as in the syllogism—

* In all these examples the middle term is enclosed in parentheses.

[Major Premiss] Every (square) IS *a parallelogram.*
[Minor Premiss] Every (square) IS *an equilateral figure,*
therefore
[Conclusion] Some equilateral figures ARE *parallelograms.*

In the fourth figure the middle term is the predicate of the major premiss and the subject of the minor, as for example,

[Major Premiss] Every triangle IS *a (plane figure).*
[Minor Premiss] Every (plane figure) IS *bounded by lines,*
therefore
[Conclusion] Some figures bounded by lines ARE *triangles.*

A syllogism consisting of three propositions, and there being four different kinds of propositions, it may be shown by the laws of combination that 64 different syllogisms (or *Moods*, as they are termed), may be formed with the same terms; thus, the major premiss may be either A, E, I, or O, and each of these may have any one again for its minor premiss, which gives 16 varieties, and each of these again may have any one of the four for its conclusion, giving in all 64 combinations.

There are, however, certain general rules or axioms relative to syllogisms, which exclude 53 of these combinations, and leave only 11 moods of the syllogism which can be legitimately employed in argument. And there are further special rules applying only to certain figures which exclude even some of these moods from being used in those figures, so that there are really only 19 legitimate modes of the syllogism. The axioms relating to syllogisms are as follows:—

1. If two terms agree with one and the same third, they agree with each other.
2. If of two terms, one agrees and the other disagrees with one and the same third, those two terms disagree with each other.
3. If neither of two terms agree with the third, those two terms may either agree or disagree with each other.

From these axioms six general rules are deduced; we shall here only state the rules, since want of space will prevent our entering into their proof.

1. *The middle term must not be taken twice particularly.*
2. *The extremes must not be taken more universally in the conclusion than in the premises.*
3. *From two negative premises no conclusion can be drawn.*
4. *A negative conclusion cannot follow from two affirmative premises.*
5. *If one of the premises be negative, the conclusion must be negative; and if one of the premises be particular, the conclusion must be particular.*
6. *From two particular premises no conclusion can be drawn.*

The special rules in the first figure are two; namely—

1. *The minor premiss must be affirmative.*
2. *The major premiss must be universal.*

In the second figure the following rules apply:—

1. *One of the premises must be negative.*
2. *The major premiss must be universal.*

In the third figure the special rules are—

1. *The minor premiss must be affirmative.*
2. *The conclusion must be particular.*

In the fourth figure there are three special rules; namely—

1. *If the major premiss be affirmative, the minor must be universal.*
2. *If the minor premiss be affirmative, the conclusion must be particular.*
3. *If either of the premises be negative, the major must be universal.*

By the application of these rules, the legitimate moods are reduced to six in each figure, or twenty-four in all; and of these five are rejected as useless, because they give only a particular conclusion when a universal one might have been drawn from the premises.

In order to impress upon the memory the nineteen allowable moods of the syllogism, logicians have contrived the following mnemonic lines, in which the three vowels denote the quantity and quality of the major premiss, the minor premiss, and the conclusion respectively. The consonants have also a meaning, which will be presently shown.

Fig. 1. Barbara, celarent, darii, ferio*que prioris.*
Fig. 2. Cesare, camestres, festino, baroko, *secundæ.*
Fig. 3. { *Tertia*, darapti, disamis, datisi, felapton, bokardo, feriso, *habet; quarta insuper addit.*
Fig. 4. Bramantip, camenes, dimaris, fesapo, fresison.

By way of illustration we shall give an example in each figure.

Bar Every (plane figure) IS *bounded by lines;*
ba Every triangle IS (*a plane figure*);
ra therefore; Every triangle IS *bounded by lines.*

Ces No circle IS (*bounded by straight lines*);
a Every square IS (*bounded by straight lines*);
re therefore; No square IS *a circle.*

Da Every (square) IS *a parallelogram;*
rap Every (square) IS *an equilateral figure;*
ti therefore; Some equilateral figures ARE *parallelograms.*

Bram Every triangle IS *a* (*plane figure*);
an Every (plane figure) IS *bounded by lines;*
tip therefore; Some figures bounded by lines ARE *triangles.*

The doctrine of the syllogism may also be derived from the maxim of Aristotle, termed "*dictum de omni et nullo*," and which may be expressed as follows; namely, *whatever may be universally affirmed or denied of any universal term, may be affirmed or denied of everything contained under it.* Now it is only to moods in the first figure that this principle can be directly and at once applied, and therefore this figure has been termed *perfect*, and the other figures which require to be reduced to the first before they can be compared with the dictum are termed *imperfect.*

The *reduction* of a syllogism is either *ostensive* or *ad impossibile*. *Ostensive reduction* consists in obtaining in the first figure either the same conclusion, or one illatively convertible into the same conclusion, by either *transposing* the premises of the original proposition so that the major may become the minor, and vice versâ, or *illatively converting* its premises. Reduction *ad impossibile* consists in substituting for the original conclusion its contradictory, and framing a new syllogism in the first figure, having for its premises this contradictory and one of the original premises, and for its conclusion the contradictory of the other original premiss.

As an example of the first mode of reduction, let us take the following syllogism in Cesare:—

Ces No circle IS *a (figure bounded by straight lines)*;
a Every square IS *a (figure bounded by straight lines)*;
re therefore; No square IS *a circle.*

And by the *simple* conversion of the major premiss, we obtain a syllogism in Celarent with the same conclusion; namely—

Ce No (figure bounded by straight lines) IS *a circle;*
la Every square IS *a (figure bounded by straight lines)*;
rent therefore; No square IS *a circle.*

Again, the following syllogism in Bramantip; namely—

Bram Every triangle IS *a (plane figure)*;
an Every (plane figure) IS *bounded by lines;*
tip therefore; Some figures bounded by lines ARE *triangles,*

may be reduced to one in Barbara with a universal conclusion by transposing the premises, thus—

Bar Every (plane figure) IS *bounded by lines;*
ba Every triangle IS *a (plane figure)*;
ra therefore; Every triangle IS *bounded by lines;*

from the conclusion of which, by conversion *per accidens*, we obtain the conclusion of the original syllogism.

As an example of the *reductio ad impossibile*, we will take the following syllogism in Baroko.

Ba Every triangle IS *a (plane figure)*;
rok Some cubes ARE NOT *(plane figures)*;
o therefore; Some cubes ARE NOT *triangles.*

Then if this conclusion is false, its contradictory must be true; namely—

Every cube IS *a triangle.*

And substituting this for the minor premiss, we obtain the following syllogism in Barbara.

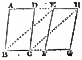
Bar Every (triangle) IS *a plane figure;*
ba Every cube IS *a* (*triangle*);
ra therefore; Every cube IS *a plane figure.*

But as this conclusion is the contradictory to the minor premiss of the original syllogism, it must be false; and since it has been correctly proved from the premises, one of those premises must be false; but the major premiss is true, therefore it is the minor which is false; and since it is the contradictory to the conclusion of the original syllogism, that conclusion is true. This mode of argument is termed the *reductio ad absurdum*, and is very frequently employed by Euclid, particularly in the demonstration of propositions from their converse.

By examining the names given to the modes, it will be observed that their initial letters are either B, C, D, or F, and these letters indicate to which of the modes in the first figure any of the modes in one of the other figures is to be reduced. Thus Cesare in the second figure is to be reduced to Celarent in the first, and Bramantip in the fourth to Barbara in the first figure. The letters *s* or *p* following a vowel imply that the proposition denoted by it is to be converted, if *s, simply*, but if *p, per accidens;* and the letter *m* signifies that the premises are to be *transposed*. The letter *k* after a vowel signifies that the proposition which it denotes is to be omitted, and the contradictory of the conclusion substituted for it, it being in fact the sign of the *reductio ad impossibile.*

Our limited space will not allow of our entering upon the subject of hypothetical and disjunctive syllogisms, or the very important one of fallacies. We must therefore conclude the foregoing brief sketch of logic with an example of one of the propositions in Euclid, formally demonstrated in syllogisms.

PROPOSITION XXXVI.



HYPOTHESIS.—If parallelograms (ABCD and EFGH) are upon equal bases and between the same parallels,

CONSEQUENCE.—They are equal to one another in area.

CONSTRUCTION.—Draw BE and CH.

DEMONSTRATION.

Syllogism 1.

Da (Things which are equal to the same) ARE *equal to one another.* [Ax. 1.]
ri The straight lines BC and EH ARE *equal to the same* FG. [Hypoth. and I. 34.]
i Therefore; The lines BC and EH ARE *equal to one another.*

Syllogism 2.

Da (Straight lines which join the adjacent extremities of two equal and parallel straight lines) ARE *themselves equal and parallel.* [I. 33.]
ri BE and CH ARE (*straight lines which join the adjacent extremities of two equal* [syl. 1] *and parallel straight lines*). [hypoth.]
i Therefore; BE and CH ARE *themselves equal and parallel.*

Syllogism 3.

Da (Parallelograms which are upon the same base and between the same parallels) ARE *equal in area.* [I. 35.]
ri ABCD and EBCH ARE (*parallelograms which are upon the same base and between the same parallels.*) [Hypoth. and syl. 2.]
i Therefore ABCD and EBCH ARE *equal in area.*

Syllogism 4.

Similar to syl. 3, proving that EFGH and EBCH ARE *equal in area.*

Syllogism 5.

Da (Things which are equal to the same) ARE *equal to one another.* [Ax. 1.]
ri ABCD and EFGH ARE (*equal in area to the same* EBCH.) [Syl. 3 and 4.]
i Therefore ABCD and EFGH ARE *equal in area to one another.*

The foregoing will sufficiently illustrate the manner in which the propositions of Euclid may be expressed in formal syllogisms, and we should recommend to the student the practice of throwing the more difficult demonstrations into the syllogistic form, as a very useful and beneficial exercise both in logic and geometry.

THE

ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

1. A Point is that which denotes position, *without possessing any magnitude.*

2. A Line is a magnitude, *having only one dimension, i.e., length.*

Corollary. The extremities of a line are points, and the intersection of one line with another is also a point.

Scholium. When a line is cut at any point, the parts of the line between that point and its extremities are termed *segments.* When the point of section (C) lies between the two extremities (A and B) of the line, the two portions into which the line is divided (AC and CB) are termed *internal* segments.

But when that point (F) lies in the production of the line beyond its extremity, the distances from the point (F) to each extremity (FD and FE) are termed *external* segments.

3. A Straight Line is a line *which lies evenly (i.e., in the same direction) between its extreme points.*

4. A Curved Line is a line *which continually changes its direction.*

Scholium. Whenever the word "line" alone is used throughout this work, it must be taken to mean a *straight line.*

5. A Surface is a magnitude, *having only two dimensions, i.e., length and breadth.*

Corollary. The extremities of a surface are lines, and the intersection of one surface with another is a line.

6. A Plane Surface, or a Plane, is a surface *which lies evenly between its extremities.*

Scholium. The terms *point, line,* and *surface,* belong to the class of *simple terms,* that is to say, they are the names given to *simple ideas,* by means of which those ideas are conveyed from one mind to another, and consequently (for the reasons stated at length in the introduction) cannot be logically defined. There must always necessarily be a certain amount of difficulty in conveying, for the first time, a simple idea from one mind to another with perfect accuracy. But the idea having been once accurately conveyed, and associated with a certain name, can at any future time be readily recalled to our mind by the mention of that name alone. The first

process, that of conveying to another mind a new simple idea, that is, a simple idea with which hitherto it had not been familiar, can only be done by a certain system of *abstraction*, that is, by presenting to it some more complex idea, in which the first simple idea is involved with others with which the mind is already familiar, by the successive abstraction of which it is left in possession of the simple idea. Let us take, as an illustration, the process by which the idea of a mathematical point would be conveyed to the mind of a person for the first time. We should first present to him the complex idea of a physical point, such as the point of a pencil or of a needle, with which he would already be familiar; we should then explain to him that the physical point involved two ideas, one of position and one of magnitude; and further, that the less the magnitude was supposed to become (or, in ordinary terms, the finer we supposed the point to be), the more precise and definite would become the position which the point occupies, and serves to mark or identify; and we should thus lead him, by the gradual abstraction of the idea of the magnitude of the point, to look upon it as infinitely small, and only to associate with it the idea of its position; and thus he would realize the idea of a mathematical point. In like manner, with a mathematical line, we should first present to him a line such as a pencil would trace on a sheet of paper, and direct his attention to the fact that the line so drawn was in reality a solid or magnitude having three dimensions, namely length, breadth, and thickness, the breadth and thickness of the black lead left by the pencil on the surface of the paper, and which constitutes the physical line presented to the eye. We should then point out the extreme minuteness of the two last dimensions of the line as compared with its length, and ask him to conceive these dimensions as becoming less and less until the idea of the line presented itself to his mind as only possessing length, but devoid of breadth and thickness.

Such, then, being the process by which the mind arrives at the true idea which such words as *point*, *line*, and *surface* represent, it would be impossible, by any formal definition of a new term, to convey for the first time, accurately, the idea which it represented. Thus, to a mind that had never heard of a mathematical point, the definition which we have given above of such a point would convey no idea. It must not, however, be therefore supposed that the foregoing definitions are useless; they serve to show the precise and limited sense in which the words are used in the following work, and so avoid the ambiguity which would arise if employed with the same latitude as in ordinary conversation. They show, for example, that the only quality of a point recognised in geometry is its position; of a line, its direction and length; and of a surface, its position and extent.

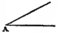
7. PARALLEL STRAIGHT LINES are straight lines in the same plane, *which, being produced to any extent in both directions, would never meet.*

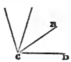
8. A RECTILINEAL ANGLE is a magnitude *formed by the inclination (i.e., opening or divergence) of two straight lines to one another, which meet in a point.*

SCHOLIUM. We have here defined an angle to be a magnitude, and we are anxious to impress the idea of its being so on the mind of the mathematical student, as it will materially assist his progress when pursuing his studies in the higher branches of analysis. In conceiving, however, an angle as a magnitude, we must be careful not to confound with it the notion of the *surface* situated between the two lines which form the angle, or to look upon its magnitude as in any way affected by the length of those lines. By way of explanation, let us borrow an illustration from the hands of a clock, and regard the angle formed by their center lines. Now at twelve

o'clock, as the hands coincide, no angle is formed by them; but from that moment they cease to coincide, and the magnitude of the angle becomes every instant greater as the minute hand moves away from the other. Now at any definite portion of time, such, for instance, as ten minutes, the hands of the clock form an angle of a certain definite magnitude, which is precisely the same whatever may be the length of those hands, whether they belong to a pocket watch or to a turret clock; in either case the interval of time is indicated by the divergence of the hands, or, in other words, by the magnitude of the angle which they form. The student has further been probably accustomed to regard the angle formed by two lines as being necessarily less than two right angles; so that if two lines were situated as in the margin, he would find it difficult to regard the angle which they formed as being that which is shaded, and would probably only be able to conceive it as the white opening BAC. Or, to revert to the hands of the clock, at three o'clock he would regard the hands as forming an angle of 90°, or the fourth part of a circle; but as the angle which they form is the measure of the angular distance that the minute hand has moved from the hour hand, we must measure that distance in the direction of its motion, and we thus find that angular distance to be 270°, or three-fourths of a circle. And by thus regarding the magnitude of an angle as the *measure* of the angular distance that a line revolving about one of its extremities has moved from its normal or first position, we are enabled to realize the idea of an angle greater even than four right angles or an entire revolution. We have only to conceive the line as moving at a uniform rate, so that the magnitude of the angle may be estimated by the length of time that it has been in motion, to see that, if the time exceeded that in which an entire revolution was performed, the angle whose magnitude it indicated had become greater than four right angles; and in like manner, after the interval required to complete two revolutions, that the angle had become greater than eight right angles, and so forth.

The point in which the two lines forming an angle meet is termed the *vertex*, and the two lines are termed the *sides*. The angle is referred to by a letter placed at the vertex, as the angle A; but if more than one angle is formed at the same point, it is then designated by three letters, one on each side and one at the vertex, the latter always being placed between the others, as the angle BCD.





The angles formed by the sides of rectilineal figures derive a variety of designations according to their relative positions and magnitudes, which will be defined in subsequent scholia.

9. A Right Angle is *half* the angle, *formed by a straight line with its continuation.*

Scholium. The line which divides the angle formed by a straight line with its continuation, into two right angles, is said to be *perpendicular* to that straight line. Thus the line CD is perpendicular to the line AB, and the angles ACD and BCD are both right angles.

10. An Obtuse Angle is an angle *which is greater than a right angle.*

11. An Acute Angle is an angle *which is less than a right angle.*

12. A Plane Figure is a plane surface *which is bounded on all sides by one or more lines.*

Scholium. The bounding line of a plane figure is termed its *perimeter,* and the space which is contained within the same is termed the *area* of the figure.

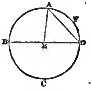
13. A Circle is a plane figure *bounded by one curved line, which is such that all straight lines drawn from it to a certain point within the figure are equal.*

14. The Circumference of a circle is the curved line *by which it is bounded.*

15. The Center of a circle is a point within the figure *equally distant from its circumference.*

16. A Radius of a circle is a straight line *drawn from the center to the circumference.*

17. A Diameter of a circle is a straight line *drawn through the center and terminated both ways by the circumference.*



Scholium. Thus the curved line ABCDF is the *circumference* of a *circle,* of which E is the *center,* BD a *diameter,* and AE a *radius.* Any portion of the circumference, as AFD, is termed an *arc* of the circle; the straight line AD joining its extremities is termed its *chord;* and the figure ADF contained by the arc and its chord is termed a *segment.* The space contained by two arcs of circles of different radii is termed a *lune,* as GHI.

18. A Rectilineal Figure is a plane surface, *bounded on all sides by straight lines.*

Scholium. The straight lines by which a rectilineal figure is bounded are termed its *sides,* which are said to *contain* the figure.

19. A Triangle is a rectilineal figure *which is bounded by three straight lines.*

Scholium. For convenience one of the lines by which a triangle is contained is termed the *base* of the triangle, the other two lines being termed its sides, and the point in which the two sides meet is termed the *vertex.*

20. An Equilateral Triangle is a triangle *which has three sides equal.*

21. An Isosceles Triangle is a triangle *which has only two sides equal.*

SCHOLIUM. When all the three sides of a triangle are unequal, it is sometimes termed a *scalene* triangle.

22. A RIGHT-ANGLED TRIANGLE is a triangle *two sides of which form a right angle.*

SCHOLIUM. In a right-angled triangle the third side opposite to the right angle is termed the *hypotenuse;* and in any triangle any side is said to *subtend* the angle opposite to it; thus the hypotenuse subtends the right angle.

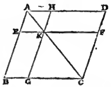
23. An OBTUSE-ANGLED TRIANGLE is a triangle *two sides of which form an obtuse angle.*

24. An ACUTE-ANGLED TRIANGLE is a triangle *the sides of which form three acute angles.*

25. A QUADRILATERAL FIGURE is a rectilineal figure *which is bounded by four straight lines.*

SCHOLIUM. A straight line drawn from any two opposite angles of a quadrilateral figure is termed a *diagonal.*

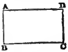
26. A PARALLELOGRAM is a quadrilateral figure *whose opposite sides are parallel.*



SCHOLIUM. If a diagonal AC be drawn to any parallelogram ABCD, and lines GH and EF be drawn respectively parallel to two contiguous sides of the same, so as to intersect in some point K of the diagonal, the parallelogram will be divided into four parallelograms, two of which, AEKH and KGCF, are said to be *about the diagonal,* and the other two of which, EBGK and HKFD, are termed the *complements* of the former.

For brevity parallelograms are frequently designated by only two letters placed at the opposite corners, as the parallelogram EG, instead of EBGK.

27. A RECTANGLE is a parallelogram *two of whose sides form a right angle.*



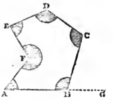
SCHOLIUM. As a rectangle is contained under four lines, two of which, AB and BC, are equal to the other two, CD and AD, it is designated as the rectangle under those two lines; thus the rectangle ABCD would be termed the *rectangle under* AB *and* BC.

28. A SQUARE is a quadrilateral figure *which has all its sides equal, and two sides of which form a right angle.*

SCHOLIUM. As a square is contained by four equal lines, upon either of which it may be conceived as constructed, it is designated as the square on one of those lines, as the *square on the line* AB.

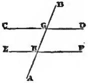
29. A Polygon is a rectilineal figure *which is bounded by more than four sides.*

Scholium. In any rectilineal figure, ABCDEF, the angles formed by its several sides on the inner side and distinguished by being shaded, are termed the *internal* or *interior angles;* when the internal angle is less than two right angles, it is termed a *salient angle*, as the angle E; but when greater, it is termed a *reëntrant angle*, as the internal angle at F. If any side is produced, the angle which its production makes with the contiguous side is called the exterior or external angle. Thus, if the side AB is produced to G, the angle CBG is termed the *external angle at* B.



When two straight lines intersect, as AB and CD, the two opposite equal angles, as AEC and DEB, are termed *vertical* angles; while those contiguous, as AEC and CEB, are termed *adjacent* angles.

When a straight line, as AB, intersects two other straight lines, as CD and EF, the angles CGH and GHF are said to be *alternate angles*, as are also DGH and GHE.



POSTULATES.

Let it be granted:—

1. That a straight line may be drawn from any point to any other point.

2. That any finite straight line may be extended, or produced, to any length.

3. That a circle may be described from any center with any radius.

Scholium. A *Postulate* is a problem the solution of which is self-evident, and therefore requiring no demonstration; it will be observed that they are only subsequently employed in the *construction* of theorems or the *solution* of problems, but never in the demonstration.

The third postulate points out the restricted use allowed to the compasses, namely, only to describe circles, but never to measure or carry distances. The compasses must be conceived as closing whenever removed from actual contact with the surface of the paper.

AXIOMS.

1. Magnitudes which are equal to the same are equal to one another.

SCHOLIUM. This axiom is frequently employed in the Elements under the form, "Magnitudes which are equal to equals are equal to one another."

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal.

5. If equals be taken from unequals, the remainders are unequal.

6. Magnitudes which are double of the same are equal to one another.

7. Magnitudes which are halves of the same are equal to one another.

SCHOLIUM. A similar extension may be given to the 6th and 7th axioms as that given above to the 1st by substituting "equals" for "the same."

8. Magnitudes which coincide with one another are equal to one another.

SCHOLIUM. The *converse* of this axiom is sometimes made use of in the elements, namely, "Magnitudes which are equal coincide with one another when similarly placed."

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

SCHOLIUM. This axiom may be otherwise expressed, namely, "If two straight lines coincide in two points, they coincide when produced."

11. All right angles are equal to one another.

SCHOLIUM. Angles being a species of magnitude, this axiom is a particular case of the 8th.

12. Through the same point two straight lines cannot be drawn parallel to the same straight line.

SCHOLIUM. This axiom is substituted for that of Euclid, not only as being more self-evident, but because the latter being the converse of the 17th proposition, it was considered that it ought to be demonstrated, which has been done after the 29th proposition.

The axioms are self-evident theorems, that is, theorems which do not admit of being demonstrated, but the truth of which is nevertheless so apparent as to be instantly admitted. No theorem should be considered as an axiom simply because it is self-evident, but only when it will not admit of being demonstrated by means of arguments founded on still simpler theorems; for it is desirable that the number of axioms should be reduced as much as possible, for which reason the 20th proposition, and some others, although as self-evident as any of the axioms, are demonstrated at length.

EXPLANATORY REMARKS.

A *proposition* in geometry is something either proposed to be solved or proved; and is consequently divided into two kinds, a problem or a theorem.

A *problem* proposes something to be done; while a *theorem* makes an assertion, of which it proposes to demonstrate the truth.

The statement of the thing to be done, or of the assertion to be proved, is termed the *enunciation* of the proposition. In the following work the enunciation is distinguished by a bolder type.

The enunciation of a problem may be divided into two parts, the *data* or things given, and the *quæsita* or things sought to be done. The former is distinguished by being printed in italics.

In like manner the enunciation of a theorem may be divided into two parts, the *hypothesis*, and the *consequence*, which it is to be proved results from that hypothesis. The former is distinguished from the consequence by being in italics.

The *solution* of a problem is the mode in which the *quæsita* are found, or the thing proposed to be done is accomplished, and is always performed by means of some other *problems* the truth of which has been either admitted or proved previously.

The *construction* of a theorem is certain things which may be required to be done by means of problems, in order to admit of the truth of the theorem being demonstrated.

The *demonstration* either of a problem or theorem is a succession of arguments logically deduced from theorems already admitted or proved, by means of which the truth of the solution of a problem or the assertion of a theorem are undeniably established.

A *lemma* is a proposition of no importance in itself, merely introduced for the purpose of demonstrating some other proposition.

A *corollary* is a proposition the truth of which immediately follows from that to which it is affixed.

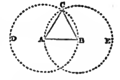
A *scholium* is a note or remark appended to any proposition by way of explanation or elucidation.

In the marginal references the following abbreviations are employed:—

Ax. 1, signifies the first axiom.
Post. 1, signifies the first postulate.
Def. 15, signifies the 15th definition.
I. 31, signifies the 31st proposition of the first book.

PROPOSITION I.

PROBLEM.—To construct an equilateral triangle upon a given *finite straight line* (AB).



(*a*) Post. 3.
(*b*) Post. 1.
(*c*) Def. 13 and 16.
(*d*) Ax. 1.

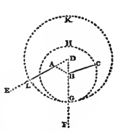
SOLUTION. *From the center* A, *at the distance* AB, *describe the circle* BCD (*a*), *and from the center* B, *at the distance* BA, *describe the circle* ACE; *and from the point* C, *in which the circles cut one another, draw the straight lines* CA, CB *to the points* A, B (*b*); *then* ABC *will be the triangle required.*

DEMONSTRATION. It is evident that the triangle ABC is constructed on the line AB. And it is also equilateral: for, the lines AC and AB being both radii of the same circle, BCD are equal (*c*), and the lines AB and CB being both radii of the same circle, ACE are equal; then, because the lines AC and CB are both equal to the same line AB, therefore they are equal to each other (*d*); that is, the three sides AB, AC, and CB are equal, *and the triangle* ABC *is therefore equilateral.*

SCHOLIUM. In this Prop. the following axiom is tacitly assumed by Euclid, viz.:—"That a circle whose center is in the circumference of another circle, must be partly within that circle, and partly without it, and therefore, that those circles must necessarily cut or intersect each other."

PROPOSITION II.

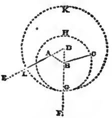
PROBLEM.—From a given *point* (A) to draw a straight line equal to a given *finite straight line* (BC).



(*a*) Post. 1.
(*b*) I. 1.
(*c*) Post. 3.
(*d*) Post. 2.
(*e*) Def. 13 and 16.

SOLUTION. *From the given point* A *draw a straight line to either extremity* B *of the given line* (*a*). *Upon* AB *construct an equilateral triangle* (*b*). *From the center* B, *at the distance* BC, *describe the circle* CGH (*c*); *and produce the straight line* DB *until it meets the circumference in* G (*d*). *From the center* D, *at the distance* DG, *describe the circle* GKL (*c*); *and produce the straight line* DA *until it meets the circumference in* L (*d*). *The straight line* AL *is equal to the given line* BC.

DEMONSTRATION. For the lines DG and DL being both radii of the same circle, GKL are equal (*e*), and if the equals DB



and DA (*f*) be taken from each respectively, the remainders BG and AL are equal (*g*); but the lines BG and BC being both radii of the same circle CGH, are equal (*e*): therefore the lines BC and AL, being both equal to the same line BG, are equal to each other (*h*). ***Therefore, from a given point* A, *a straight line has been drawn equal to a given straight line* BC.**

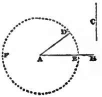
(*e*) Def. 13 and 16.
(*f*) Construction.
(*g*) Ax. 3.
(*h*) Ax. 1.

Scholia. 1. The construction of this problem will somewhat vary according to the relative positions of the point A and the line BC.

2. In practice this problem will be solved by measuring the length of the given line BC with a pair of compasses; and then, applying one leg of the compasses to the point A, the other leg will mark the length of the line required to be drawn from A. In geometry, however, such use of the compasses is not permitted. The only way in which they may be employed is that allowed in the third postulate, viz. to describe a circle whose circumference shall pass through a given point about some other given point as a center. The compasses must be supposed to close of themselves whenever removed from the paper, so that no distance can be *carried* by means of them. This restricted use of the compasses being borne in mind will enable the student to see the necessity of the first three problems in this book.

PROPOSITION III.

Problem.—From the greater of *two* given *straight lines* (AB and C) to cut off a part equal to the less.



Solution. *From either extremity* A *of the greater given line draw a straight line* AD *equal to the lesser given line* C (*a*). *From the center* A, *at the distance* AD, *describe the circle* DEF (*b*), *which shall cut off* AE *equal to the lesser line* C.

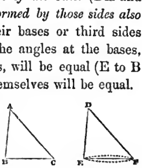
Demonstration. For the lines AD and AE being both radii of the same circle DEF are equal (*c*). But AD and C are equal (*d*); therefore, because AE and C are both equal to the same line AD, they are equal to each other (*e*); *and from* AB *the greater of two given lines, a part* AE *has been cut off, equal to* C *the less.*

(*a*) I. 2.
(*b*) Post. 3.
(*c*) Def. 13 and 16.
(*d*) Const.
(*e*) Ax. 1.

Scholium. By a similar operation the lesser line could be extended to equal the greater; thus from either extremity of the lesser line let a line be drawn equal to the greater, then about this same extremity as a center describe a circle with a radius equal to the greater line; extend the lesser line to meet the circumference of this circle, and it will equal the greater line.

PROPOSITION IV.

THEOREM. *If two triangles* (ABC and DEF) *have two sides of the one respectively equal to two sides of the other* (DE and DF to AB and AC), *and the angles formed by those sides also equal to one another* (D to A); [1] their bases or third sides (EF and BC) will be equal; [2] and the angles at the bases, which are opposite to the equal sides, will be equal (E to B and F to C); [3] and the triangles themselves will be equal.



(a) Ax. 10.
(b) Scholium 1.
(c) Ax. 8.

DEMONSTRATION. For, if the triangle ABC be applied to DEF, so that the point A may be on the point D, the point B on the straight line DE, and that AC and DF may lie on the same side; then AB must lie wholly on DE, for otherwise two straight lines would enclose a space (*a*); and because AB is equal to DE, the point B must coincide with the point E (*b*). Further, because the angles A and D are equal, the side AC must fall upon the side DF; and because AC is equal to DF, the point C must coincide with the point F.

[1.] Therefore, as the points B and C coincide with the points E and F, *the base* BC *must coincide with the base* EF, *and be equal to it* (*c*); for otherwise two straight lines would enclose a space (*a*).

[2.] And as the sides which form the angles B and C coincide with the sides which form the angles E and F, *those angles themselves must coincide, and therefore must be equal* (*c*).

[3.] And as the straight lines which contain the triangle ABC coincide with and are equal to the straight lines which contain the triangle DEF, *therefore the triangles themselves must coincide, and must therefore be equal* (*c*).

SCHOLIA. 1. In the above demonstration (at *b*) an axiom is assumed the converse of the eighth axiom, namely, "Magnitudes which are equal coincide with one another when similarly placed."

2. In every triangle there are six quantities or magnitudes, namely, the three sides and the three angles; and (except in two particular cases) when any three of these are given, the other three can be found and the triangle determined. If, therefore, two triangles are found to agree in any three of those quantities by which the triangles are determined, it is evident that those triangles must be equal. The following are the only six cases which can occur:—

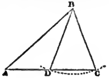
1. The three angles.
2. The three sides.
3. Two sides and the angle between them.
4. Two sides and the angle opposite to one of them.
5. Two angles and the side between them.
6. Two angles and the side opposite to one of them.

The first case is one of the two in which the triangle is not determined; for a triangle may have its sides increased or diminished to any extent without altering the magnitude of its angles.

The second case is demonstrated in the eighth proposition.

The third case is the subject of the present proposition.

The fourth case is the other one in which the triangle is not determined; for it is quite possible to have two triangles having two sides of the one equal to two sides of the other, and one of the opposite angles of the one equal to the similar angle of the other, and yet for the triangles themselves not to be equal. Thus, let ABC be a triangle in which neither A nor C are right angles and A is less than C; then from B as a center, and the distance BC as radius, describe a circle cutting AC in D, and draw DB. Now it is evident that, in the triangles ABC and ABD, we have the two sides AB and BC equal to the two AB and BD, and the opposite angle A the same in both; and yet the two triangles are not equal.



The fifth and sixth cases are demonstrated in the twenty-sixth proposition.

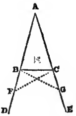
3. The application of one figure to another, so as to prove or disprove their coincidence, as made use of in this proposition, is termed *superposition*. It has been objected to by some mathematicians as not being strictly geometrical; but, as Mr. De Morgan has observed, it requires only the admission of the following postulate, "That any figure may be removed from place to place without alteration of form, *and a plane figure may be turned over on the plane.*" The latter portion, printed in italics, is required for the proof of the fifth proposition.

4. The enunciation of this proposition really includes three distinct propositions, which have been distinguished by separate numbers. The demonstration of the first of these is an example of the negative or indirect proof termed "Reductio ad Absurdum," which consists in proving a proposition by showing that if it is denied an obvious absurdity follows. Concerning this method of proof, see the Introduction.

5. The *base* of a triangle is the third side of a triangle as distinguished from the other two, without any regard to whether the triangle stands upon it or not.

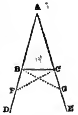
PROPOSITION V.

THEOREM.—[1] *If a triangle* (ABC) *be isosceles,* the angles at the base (ABC and ACB) are equal to one another; [2] *and if the equal sides be produced,* the angles formed by the produced sides and the base below the same (CBD and BCE) shall be equal.



CONSTRUCTION. *Produce the equal sides* AB *and* AC (*a*), *and in the produced part of one of them* AB *take any point* F, *and from the other cut off* AG *equal to* AF (*b*). *Draw a straight line from* C *to* F, *and from* B *to* G (*c*).

(*a*) Post. 2.
(*b*) I. 3.
(*c*) Post. 1.

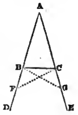


DEMONSTRATION. [2.] In the triangles ACF and ABG, the side AC is equal to the side AB (*d*), the side AF to the side AG (*e*), and the angle A is common to both; therefore the base CF is equal to the base BG, the angle ACF to the angle ABG, and the angle F to the angle G (*f*); then taking the equal lines AC and AB from the equal lines AG and AF, the remainders CG and BF are equal (*g*); therefore in the triangles BCG and CBF, because the side CG is equal to the side BF, the side BG to the side CF, and the angle G to the angle F, *the angle* BCG *must be equal to the angle* CBF (*f*), *which are the angles formed by the produced sides and the base, below the same.*

(*d*) Hypoth.
(*e*) Constr.
(*f*) I. 4.
(*g*) Ax. 3.

[1.] Further in the same triangles the remaining angles must be equal, BCF to CBG (*f*); and if these equal angles be taken from the equal angles ACF and ABG, *the remaining angles,* ACB *and* ABC, *will be equal* (*g*), *which are the angles at the base of the given triangle.*

COROLLARY. Hence every *equilateral* triangle is also equiangular; for if each side be taken in succession as the base, it may be shown that the angles adjacent to the side so taken are equal.



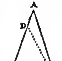
SCHOLIA. 1. This proposition may also be proved in the following manner: if the triangle ABC be turned over on the plane (see scholium 3 to the preceding proposition), so that the position of the point A may be unaltered, while the side AB lies on AC, then, since the angle A is the same in both, the side AC must fall on AB; and because the sides AB and AC are equal (*a*), the point C will coincide with the point B, the point B with C, the angle ACB with the angle ABC, and the angle BCG with the angle CBF; and therefore [1] the angle ACB will be equal to the angle ABC (*b*), and [2] the angle BCG equal to the angle CBF (*b*).

2. The enunciation of this proposition really includes two separate propositions which have been distinguished by numbers.

(*a*) Hypoth.
(*b*) Ax. 8.

PROPOSITION VI.

THEOREM.—*If two angles* (B and C) *of a triangle* (ABC) *are equal,* the sides (AC and AB) opposite to those angles are also equal.



DEMONSTRATION. For if AB be not equal to AC, one of them is greater than the other; let AB be the greater, from it cut off DB equal to AC the less (*a*), and draw the line DC. Then in the triangles ABC and DBC, because the side AC is equal to the side DB, the base BC common to both, and the angle ACB equal to the angle DBC (*b*), therefore the tri-

(*a*) I. 3.
(*b*) Hypoth.
(*c*) I. 4.

angles themselves are equal (*c*); the greater ABC to the lesser DBC, which is absurd; *therefore neither of the sides* AC *or* AB *being greater than the other, they are equal.*

COROLLARY. Hence every *equiangular* triangle is also equilateral, which may be shown by taking each side in succession as the base.

SCHOLIA. 1. It should be observed that the portion equal to the lesser side must be cut off from that end of AB next to the equal angle; otherwise no proof can be drawn from proposition iv.

2. This proposition is the *converse* of the first part of the preceding; that is to say, the hypothesis of one is the predicate in the other, and *vice versâ*, which will be immediately seen by expressing them as under:—

PROP. V. *If two sides are equal*, the opposite angles are equal.

PROP. VI. *If two angles are equal*, the opposite sides are equal.

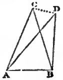
The truth of a proposition does not establish that of its converse, for the reasons explained in treating of the conversion of propositions in the Introduction.

This proposition is demonstrated by the method of "reductio ad absurdum," which will be found, in the Elements, to be most frequently employed in the demonstration of *converse* propositions.

PROPOSITION VII.

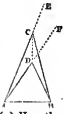
THEOREM.—*If two triangles* (ABC and ABD) *be upon the same base* (AB) *and on the same side of it*, they cannot have their sides which are terminated in one extremity of that base equal to one another, and also those which are terminated in the other extremity (AC to AD and BC to BD).

DEMONSTRATION. If it be possible, let there be two triangles on the line AB; then must either [1] the vertex of each of the triangles be without the other one, or [2] the vertex of one triangle within the other one, or [3] upon one side of it.



[1.] Let the vertex of each triangle be without the other one. *Draw a straight line from* C *to* D, *the two vertices.* Then because in the triangle BCD the sides BC and BD are equal (*a*), therefore the angles BDC and BCD are equal (*b*). Also, because in the triangle ACD the sides AC and AD are equal (*a*), therefore the angles ADC and ACD are equal (*b*). Now the angle ACD is greater than BCD (*c*); and because ACD and ADC are equal, therefore is ADC also greater than BCD; and as BDC is greater than ADC (*c*), therefore BDC is greater than BCD; but they have already been proved to be equal, which is absurd.

(*a*) Hypoth.
(*b*) I. 5.
(*c*) Ax. 9.



(a) Hypoth.
(b) I. 5.
(c) Ax. 9.

[2.] Let the vertex of one triangle be within the other one. *Produce the sides* AC and AD, *and draw a line from* C to D. Then, because in the triangle BCD the sides BC and BD are equal (a), therefore the angles BDC and BCD are equal (b). Also, because in the triangle ACD the two sides AC and AD are equal (a), therefore the angles ECD and FDC, on the other side of the base, are equal (b). Now the angle ECD is greater than the angle BCD (c); and because FDC and ECD are equal, therefore is FDC also greater than BCD; and as BDC is greater than FDC (c), therefore is BDC greater than BCD; but they have already been proved to be equal, which is absurd.

[3.] Let the vertex D of one triangle fall on the side AC of the other. Then it is evident that the sides BC and BD are not equal, which is contrary to the hypothesis.

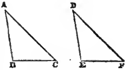
Therefore in no case can two triangles be upon the same base, and upon the same side of it, that have their sides which are terminated in one extremity of that base equal to one another, and also those which are terminated in the other extremity.

SCHOLIA. 1. The only use made of this proposition is to prove that which follows it, which can, however, be demonstrated without it, as is done in the scholium attached to that proposition.

2. The form of argument adopted by Euclid in the demonstration of this proposition is that termed a *dilemma*, and this is the only instance throughout the Elements in which it is employed. The argument will also be seen to be by the "reductio ad absurdum."

PROPOSITION VIII.

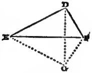
THEOREM.—*If two triangles* (ABC and DEF) *have two sides of the one respectively equal to two sides of the other* (AB and AC to DE and DF), *and have also their bases equal* (BC to EF), then the angles (A and D) formed by the equal sides are equal.



(a) I. 7.
(b) Ax. 8.

DEMONSTRATION. For if the triangle ABC be applied to the triangle DEF, so that their equal bases may coincide, and that the two triangles may lie on the same side, their equal sides must coincide (a), BA with ED, and CA with FD, *and therefore the angles* A *and* D *must coincide, and be equal to each other* (b).

Corollary. Hence also the angles opposite the equal sides are equal, B to E and C to F; and also the triangles themselves are equal.



Scholia. 1. This proposition may be demonstrated in the following manner, without any reference to the seventh. Let the triangle ABC be applied to the triangle DEF, so that their bases may coincide, and that the two triangles may lie on opposite sides; join the vertices DG. Then, because in the triangle DEG the two sides ED and EG are equal (*a*), therefore the angles EDG and EGD are equal (*b*). Also, because in the triangle DFG the two sides FD and FG are equal (*a*), therefore the angles FDG and FGD are equal (*b*). Then if the equal angles EDG and EGD be added to the equal angles FDG and FGD, the whole angles EDF and EGF will be equal (*c*). But the angle EGF is equal to BAC (*a*), *therefore the angles* EDF *and* BAC *are equal* (*d*).

(*a*) Hypoth.
(*b*) I. 5.
(*c*) Ax. 2.
(*d*) Ax. 1.

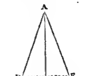
2. This proposition is the *converse* of proposition iv. When a theorem has several hypotheses and one predicate, if another theorem be framed having one of those hypotheses for its predicate and the predicate of the first as one of its hypotheses, the two theorems are the converse of each other; and it is in this sense that the eighth proposition is the converse of the fourth, as will be immediately seen by expressing them in the following manner:—

Prop. IV.—If *two sides are equal*, and $\left\{\begin{array}{l}\textit{the angles opposite}\\ \textit{the bases are equal,}\end{array}\right\}$ then the bases are equal.

Prop. VIII.—If $\left.\begin{array}{l}\textit{two sides are equal}\\ \text{and } \textit{the bases are equal,}\end{array}\right\}$ then $\left\{\begin{array}{l}\text{the angles opposite}\\ \text{the bases are equal.}\end{array}\right.$

PROPOSITION IX.

Problem.—To bisect a given *rectilineal angle* (BAC).



Solution. *Take any point* D *in* AB, *and from* AC *cut off* AE *equal to* AD (*a*); *draw* DE, *and upon the side furthest from* A *construct an equilateral triangle* DEF (*b*); *then a straight line drawn from* A *to* F *will bisect the angle* BAC.

Demonstration. Because in the triangles AFD and AFE, the sides AD and AE are equal (*c*), the side AF common to both, and the base DF equal to the base EF (*c*); therefore the angle DAF is equal to the angle EAF (*d*), *and the given rectilineal angle* BAC *is bisected by the straight line* AF.

(*a*) I. 3.
(*b*) I. 1.
(*c*) Solution.
(*d*) I. 8.

Scholia. 1. The direction to construct the equilateral triangle on the side of DE furthest from A is given in order to avoid the possibility of the vertex falling on the point A, in which case the line AF could not be drawn.

2. By a repetition of this problem, an angle may be divided into

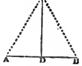
4, 8, 16, &c., equal parts; that is, into any number of equal parts which can be expressed by a power of 2.

3. The *trisection* of a given angle, or its division into *three* equal parts, is a problem that has baffled the attempts of the most able mathematicians, and can only be solved in particular cases. When the given angle is a right angle, or such an angle as is obtained by the division of a right angle by any power of 2, it may be solved in the manner shown in the sixth corollary of proposition xxxii.

PROPOSITION X.

PROBLEM.—To bisect a given *finite straight line* (AB).

SOLUTION. *Construct upon it an equilateral triangle* ABC (*a*), *and bisect the angle* ACB *by the straight line* CD (*b*), *which will also bisect the given line in the point* D.



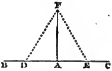
DEMONSTRATION. Because in the triangles ACD and BCD the sides AC and BC are equal (*c*), the side CD is common to both, and the angle ACD equal to the angle BCD (*c*), therefore the base AD is equal to the base DB (*d*), *and the straight line* AB *is bisected in the point* D.

(*a*) I. 1.
(*b*) I. 9.
(*c*) Solution.
(*d*) I. 4.

PROPOSITION XI.

PROBLEM.—From a given *point* (A) in a given *straight line* (BC) to draw a perpendicular to that line.

SOLUTION. *Take any point* D *in* AB, *and from* AC *cut off* AE *equal to* AD (*a*); *and upon* DE *construct an equilateral triangle* DEF (*b*); *then a straight line drawn from* A *to* F *will be perpendicular to the given line.*



DEMONSTRATION. Because in the triangles AFD and AFE, the sides AD and AE are equal (*c*), the side AF common to both, and the base DF equal to the base EF (*c*); therefore the angle DAF is equal to the angle EAF (*d*), and the angle formed by the straight line AB with its continuation AC is bisected by the straight line AF, *which is therefore perpendicular to the given line* (*e*).

(*a*) I. 3.
(*b*) I. 1.
(*c*) Solution.
(*d*) I. 8.
(*e*) Def. 9.

COROLLARY. By help of this problem it may be demonstrated that "*If two lines be straight,* they cannot have a common segment."

(a) I. 11.
(b) Def. 9.
(c) Ax. 11.
(d) Ax. 1.

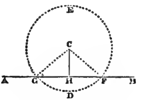
DEMONSTRATION. If it be possible, let the segment AB be common to the two straight lines BC and BD. *From the point* B *draw* EB *perpendicular to* AB (*a*). Then, because BC is the continuation of the straight line AB, the angles ABE and EBC will both be right angles (*b*) and will be equal (*c*). Also, because BD is the continuation of the straight line AB, the angles ABE and EBD will both be right angles (*b*) and will be equal (*c*). Therefore the angle EBD is equal to the angle EBC (*d*), the less to the greater, which is absurd; *therefore two straight lines cannot have a common segment.*

SCHOLIA. 1. By this proposition a perpendicular can be drawn at the extremity of a given line, by first producing the line.

2. By comparing the foregoing with proposition ix. it will be immediately seen that the former is only a particular case of the latter. Proposition ix. is to bisect *any* given angle, and proposition xi. is to bisect *that particular* angle which a straight line forms with its continuation. The letters in the diagram used by Euclid have, in this particular instance, been deviated from, in order to make the similarity between these two propositions more apparent.

PROPOSITION XII.

PROBLEM.—To draw a straight line perpendicular to a given *straight line of an unlimited length* (AB), from a given *point* (C) *without it.*



(a) I. 10.
(b) Solution.
(c) Def. 13 and 16.
(d) I. 8.
(e) Def. 9.

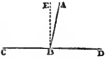
SOLUTION. *Take any point* D *upon the other side of* AB, *and from the center* C, *at the distance* CD, *describe a circle cutting the given line in* F *and* G. *Bisect* FG *in* H (*a*), *and from the given point* C *draw the straight line* CH; *it will be perpendicular to the given line* AB.

DEMONSTRATION. *Draw the lines* CF *and* CG. Then, because in the triangles CHF and CHG, the sides FH and GH are equal (*b*), the side CH common to both, and the base CF equal to the base CG (*c*), therefore the angle CHF is equal to the angle CHG (*d*), and therefore the line CH is drawn perpendicular to the given line AB (*e*).

SCHOLIUM. In this proposition it is *assumed* that the given line AB will be cut by the circle in *two* points. This will be evident if we consider that a portion of the circumference of the circle lies on each side of the line AB, and that as the circumference is a continuous line it must necessarily cross the line twice. The given line is supposed to be *unlimited* in length, because otherwise it might so happen that the circle described from C might not cut it at all.

PROPOSITION XIII.

THEOREM.—*If a straight line* (AB) *standing upon another* (CD) *forms angles with it,* they are either two right angles, or are together equal to two right angles.



DEMONSTRATION. For if the line AB is perpendicular to CD, the angles ABC and ABD are two right angles (*a*). But if not, *draw* BE *perpendicular to* CD (*b*), and it is evident that the angles ABD and ABC are together equal to the angles EBD and EBC, *and therefore to two right angles.*

(*a*) Def. 9.
(*b*) I. 11.

COROLLARY 1. From this proposition it is evident, that *if several straight lines stand on the same side of another straight line at the same point, and make angles with it,* all those angles are together equal to two right angles.

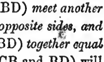
COROLLARY 2. Also, *if two straight lines intersect,* the four angles which they form at the point of intersection are together equal to four right angles.

COROLLARY 3. And consequently, *if any number of straight lines diverge from the same point,* all the angles which they make taken together are equal to four right angles.

SCHOLIUM. It is necessary, in the enunciation of this proposition, to insert the words "forms angles with it," to exclude the case in which the line AB stands at either extremity of CD.

PROPOSITION XIV.

THEOREM.—*If two straight lines* (CB and BD) *meet another straight line* (AB) *at the same point and on opposite sides, and make the adjacent angles with it* (ABC and ABD) *together equal to two right angles,* those two straight lines (CB and BD) will form one continued straight line.

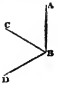


DEMONSTRATION. For, if possible, let BE, and not BD, be the continuation of the straight line CB. Then ABC and ABE are together equal to two right angles (*a*), and because ABC and ABD are together equal to two right angles (*b*), therefore ABC and ABE taken together are equal to ABC and ABD taken together (*c*). If now from these equals we take away the angle ABC, which is common to both, the remaining angles ABE and ABD shall be equal (*d*) a part to the whole, which is absurd; therefore

(*a*) I. 13.
(*b*) Hypoth.
(*c*) Ax. 1.
(*d*) Ax. 3.

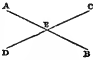
BE is not the continuation of CB. And in like manner it may be proved that no other straight line except BD can be the continuation of CB; *therefore* BD *and* CB *form one continued straight line.*

SCHOLIUM. The above proposition is proved by the "reductio ad absurdum." It is necessary that the two straight lines CB and BD should be on *opposite* sides of AB; for otherwise they might form angles with it together equal to two right angles, without being in the same continued straight line, as in the annexed figure.



PROPOSITION XV.

THEOREM.—*If two straight lines* (AB and CD) *intersect one another*, the vertical angles are equal (CEA to DEB, and CEB to AED).



DEMONSTRATION. Because the straight line AE forms with CD the angles CEA and AED, they are together equal to two right angles (*a*); also because the straight line DE forms with AB the angles AED and DEB, they are together equal to two right angles (*a*); therefore CEA and AED taken together are equal to AED and DEB taken together (*b*). If now from these equals we take away the angle AED, which is common to both, the remaining *angles* CEA *and* DEB *shall be equal* (*c*). And in like manner it can be proved that the angles CEB and AED are equal.

(*a*) I. 13.
(*b*) Ax. 1.
(*c*) Ax. 3.

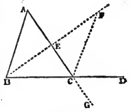
SCHOLIA. 1. The second and third corollaries to proposition xiii. may also be drawn as corollaries from the present one.

2. The *converse* of this proposition may be demonstrated as follows. *If four straight lines meet in the same point, and make the vertical angles equal,* each alternate pair of lines will form one continued straight line. For since CEA is equal to BED (*a*), and CEB is equal to AED (*a*), therefore CEA and CEB taken together are equal to BED and AED taken together (*b*); and because the whole four taken together are equal to four right angles (*c*), therefore CEA and CEB taken together are equal to two right angles, and consequently AE *and* EB *form one continued straight line* (*d*). And in like manner it may be shown *that* CE *and* ED *also form one continued straight line.*

(*a*) Hypoth.
(*b*) Ax. 2.
(*c*) I. 13, cor. 3.
(*d*) I. 14.

PROPOSITION XVI.

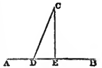
THEOREM.—*If one side* (BC) *of a triangle* (ABC) *be produced,* the external angle (ACD) is greater than either of the internal opposite angles (ABC or A).



CONSTRUCTION. *Bisect* AC *in* E (*a*), *draw* BE, *and produce it until* EF *is equal to* BE. *Also join* FC.

DEMONSTRATION. Because in the triangles EAB and ECF the side EA is equal to the side EC (*b*), the side EB to the side EF (*b*), and the angle AEB is equal to the angle FEC (*c*), therefore the angle A is equal to the angle ECF (*d*). And therefore ACD being greater than ECF, *is also greater than* A. In like manner, if the side AC be produced, it may be proved that the angle BCG, and therefore its equal ACD (*c*), *is greater than the angle* ABC.

(*a*) I. 10.
(*b*) Constr.
(*c*) I. 15.
(*d*) I. 4.



COROLLARY 1. *If from any point* (C) *two straight lines be drawn to meet a third straight line* (AB), *one of them* (CD) *perpendicular to it, and the other* (CE) *not;* then that which is perpendicular shall be on that side of the other on which it forms an acute angle.

For if it be possible, let it fall on the same side as the obtuse angle CEA; then the angle CDA, being a right angle, is less than CEA (*a*); but CDA is also greater than the internal opposite angle CEA (*b*), which is absurd; therefore CD cannot fall on the side next the obtuse angle, *but on the same side as the acute angle.*

(*a*) Def. 10.
(*b*) I. 16.

COROLLARY 2. *If two straight lines be drawn from any point* (C) *to the same straight line* (AB), they cannot both be perpendicular to it.

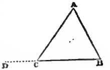
For if it be possible, let both CD and CE be perpendicular to AB; then the angle ADC is equal to AEC (*a*); but ADC is also greater than AEC (*b*), which is absurd; therefore *the straight lines* CD *and* CE *cannot both be perpendicular to* AB.

(*a*) Ax. 11.
(*b*) I. 16.

SCHOLIUM. Both the sixteenth and seventeenth propositions are included in the thirty-second, and as they are not employed until subsequent to that proposition, they might have been omitted in this place.

PROPOSITION XVII.

THEOREM.—*If any two angles are those of a triangle* (ABC), they are together less than two right angles.

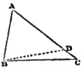


(a) I. 16.
(b) I. 13.

DEMONSTRATION. *Produce* BC *to* D. Then the external angle ACD is greater than the internal opposite angle B (*a*); and if to both the angle ACB be added, ACD and ACB taken together are greater than B and ACB taken together; but ACD and ACB taken together are equal to two right angles (*b*), therefore B *and* ACB, *two of the angles of the triangle* ABC, *are together less than two right angles.* And in like manner it may be proved that *either the angles* A *and* B *or* A *and* ACB *are together less than two right angles.*

PROPOSITION XVIII.

THEOREM.—*If one side* (AC) *of any triangle* (ABC) *be greater than another* (AB), the angle (ABC) opposite to the greater side is greater than the angle (C) which is opposite to the less.



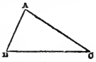
(a) I. 3.
(b) Constr.
(c) I. 5.
(d) I. 16.

CONSTRUCTION. *From the greater side* AC *cut off* AD *equal to the less* AB (*a*), *and join* BD.

DEMONSTRATION. Because in the triangle ABD the sides AD and AB are equal (*b*), therefore the angles ADB and ABD are equal (*c*). But the angle ADB, being the external angle of the triangle BCD, is greater than the internal opposite angle C (*d*), therefore the angle ABD is greater than the angle C; and as ABC is greater than ABD, therefore ABC, *the angle opposite the greater side, is greater than* C, *the angle opposite the less.*

PROPOSITION XIX.

THEOREM.—*If in any triangle* (ABC) *one angle* (B) *is greater than another* (C), the side (AC) which is opposite to the greater angle is greater than the side (AB) which is opposite to the less.



(a) I. 5.
(b) I. 18.

DEMONSTRATION. For the side AC must either be equal to, less than, or greater than AB.

It is not equal to AB, because then the angle B would be equal to the angle C (*a*), while by the

hypothesis, B is greater than C. And it is not less than AB, because then the angle B would be less than C (*b*); therefore *the side* AC *opposite to the greater angle is greater than the side* AB *opposite to the less.*

SCHOLIA. 1. This proposition is the converse of the preceding proposition, and bears the same relation to proposition vi. that the preceding proposition does to proposition v. This relation is seen by combining them in the following manner. One *angle* of a triangle is greater or less than another (prop. xviii.), or equal to it (prop. v.), according as the side opposed to the one is greater than, less than, or equal to the side opposed to the other. And one *side* of a triangle is greater or less than another (prop. xix.), or equal to it (prop. vi.), according as the angle opposed to the one is greater than, less than, or equal to the angle opposed to the other.

The mutual relation of these four propositions may also be shown in the following manner:—

$\begin{cases} \text{Prop. V.—} \textit{If } AB = AC, \text{ then } \angle C = \angle B. \\ \text{Prop. VI.—} \textit{If } \angle C = \angle B, \text{ then } AB = AC. \end{cases}$

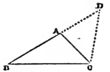
$\begin{cases} \text{Prop. XVIII.—} \textit{If } AB > AC, \text{ then } \angle C > \angle B. \\ \text{Prop. XIX.—} \textit{If } \angle C > \angle B, \text{ then } AB > AC. \end{cases}$

The propositions connected by a bracket are the *converse* of each other, because that which is the hypothesis in the one is the predicate in the other.

2. The form of argument employed in this proposition is the "reductio ad absurdum."

PROPOSITION XX.

THEOREM.—*Any two sides* (BA and AC) *of a triangle* (ABC) are together greater than the third side (BC).



CONSTRUCTION. *Produce either of the two sides, as* BA, *until* AD *is equal to* AC, *the other side, and join* CD.

DEMONSTRATION. Because the triangle ACD is isosceles (*a*), the angles ACD and D are equal (*b*). But the angle BCD is greater than ACD (*c*), therefore it is greater than the angle D. Now because in the triangle BCD the angle BCD is greater than the angle D, therefore the side BD opposite BCD is greater than the side BC opposite D (*d*). But BD is equal to BA and AC (*a*), therefore *the sides* BA *and* AC *are together greater than the third side* BC.

(*a*) Constr.
(*b*) I. 5.
(*c*) Ax. 9.
(*d*) I. 19.

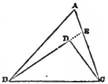
COROLLARY. It follows from this proposition that the difference between any two sides of a triangle is less than the third side; for since BA and AC are together greater than BC, if AC be subtracted from each, BA will be greater than the difference between BC and AC.

SCHOLIA. 1. By Archimedes and some others this proposition was considered to be an axiom, or self-evident. The self-evidence, however, of a proposition is not a sufficient reason for its being placed amongst the axioms, the number of which should always be kept as small as possible; and no proposition should ever be admitted as an axiom if it can be demonstrated.

2. In this proposition, and many others in the Elements, an axiom is assumed which is an extension of the fifth, namely, "if one quantity be greater than another, and equals be taken from both, the remainder of the one is greater than the remainder of the other."

PROPOSITION XXI.

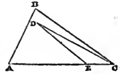
THEOREM.—*If from a point* (D) *within a triangle* (ABC) *two straight lines* (BD and DC) *be drawn to the extremities of any side* (BC), [1] they are together less than the sum of the two other sides of the triangle (BA and AC), [2] and they form a greater angle.



(a) I. 20.
(b) I. 16.

DEMONSTRATION. [1.] *Produce* BD *to* E. In the triangle ABE the two sides BA and AE are greater than the third side BE (a), to each of these add EC, then the sides BA and AC are greater than BE and EC. Again, in the triangle CED, the two sides DE and EC are greater than the third side DC (a), to each of these add BD, then the sides BE and EC are greater than BD and DC. But BA and AC are greater than BE and EC, therefore *the sides* BA *and* AC *are greater than the straight lines* BD *and* DC.

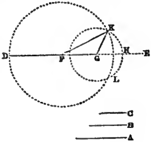
[2.] Because in the triangle CED the external angle BDC is greater than the internal opposite angle BEC (b), and because in the triangle ABE the external angle BEC is greater than the internal opposite angle A (b), therefore *the angle* BDC *formed by the two straight lines is greater than the angle* A *formed by the two sides of the triangle.*



SCHOLIUM. If the two straight lines are not drawn to the *extremities* of the base, it is possible for them to exceed the two sides of the triangle in any ratio less than that of 2 to 1; for if the lines be drawn, as DC and DE in the figure, they may easily be shown to be greater than the two sides AB and BC.

PROPOSITION XXII.

PROBLEM.—Given *three finite straight lines* (AB and C) *of which any two together are greater than the third*, to construct a triangle whose sides shall be respectively equal to the given lines.



SOLUTION. *Draw a straight line* DF *equal to* A, *and on the continuation of this line take* FG *equal to* B, *and* GH *equal to* C (*a*). *From the center* F *at the distance* DF *describe a circle* (*b*), *and from the center* G *at the distance* GH *describe another circle* (*b*); *from the point of intersection* K *draw* KF *and* KG; *then will the triangle* FKG *have its sides equal to the three given lines.*

DEMONSTRATION. For DF and FK, being both radii of the same circle, are equal to the same line A (*c*); also GK and GH, being both radii of the same circle, are equal to the same line C (*c*); and FG is equal to B (*d*); therefore *the three sides* FK, FG, *and* GK *of the triangle* FKG *are respectively equal to the given straight lines* A, B, *and* C.

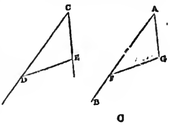
(*a*) I. 3.
(*b*) Post. 3.
(*c*) Def. 15 and 16.
(*d*) Solution.

SCHOLIA. 1. In the above demonstration it is *assumed* that the two circles must intersect in the point K, whereas it should have been demonstrated that such is the case, which may be done in the following manner. Because the sum of the radii FK and GK is greater than FG, therefore a part of each circle struck with those radii must be within the other; and because the sum of FG and GK is greater than FK, therefore a portion of each circle is without the other; and therefore their circumferences must cut in some point as K.

2. If the three given lines A, B, and C be equal to each other, this proposition becomes identical with the first one, as will be evident on comparing them together.

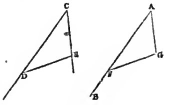
PROPOSITION XXIII.

PROBLEM.—At a given *point* (A) in a given *straight line* (AB) to form a rectilineal angle equal to a given *rectilineal angle* (C).



SOLUTION. *In the two sides of the given angle take any points* D *and* E; *draw* DE, *and construct the triangle* AFG, *whose side* AF *shall be equal to* CD *and shall have one extremity coincide with the given*

point A, *and the other on the given line* AB, *and whose other two sides* AG *and* FG *shall be respectively equal to* CE *and* DE (*a*); *then shall the angle* A *be equal to the angle* C.

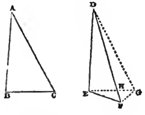


DEMONSTRATION. For as the triangles CDE and AFG have the sides CD and CE equal to AF and AG, and the bases DE and FG equal, *the angles* C *and* A *formed by the equal sides are equal* (*b*).

(*a*) I. 22.
(*b*) I. 8.

PROPOSITION XXIV.

THEOREM.—*If two triangles* (ABC and DEF) *have two sides of the one respectively equal to two sides of the other* (AB and AC to DE and DF), *and if the angle* (A) *formed by the two sides of one of them be greater than the angle* (EDF) *formed by the two sides equal to them of the other,* the side (BC) opposite to that greater angle is greater than the side (EF) which is opposite to the less.



CONSTRUCTION. *From the point* D *draw the straight line* DG, *making, with the side* DE, *which is not the greater, an angle* EDG *equal to the angle* A (*a*); *make* DG *equal to* AC (*b*), *and draw* EG *and* FG.

DEMONSTRATION. Because in the triangles ABC and DEG the sides AB and AC are equal to DE and DG, and the angle A is equal to EDG, therefore BC is equal to EG (*c*). And because DG and DF are equal (*d*), and DH is less than DG (*e*), the point F must be below the base EG; therefore the angle DGF is greater than the angle EGF (*f*); but because in the triangle DFG the sides DG and DF are equal (*d*), therefore the angle DGF is equal to DFG (*g*); but DGF is greater than EGF (*f*), therefore DFG is greater than EGF; and because EFG is greater than DFG (*f*), therefore is EFG greater than EGF. Then in the triangle EGF the angle EFG is greater than EGF, therefore the side EG is greater

(*a*) I. 23.
(*b*) I. 3.
(*c*) I. 4.
(*d*) Hypoth. and Constr.
(*e*) Scholium 1.
(*f*) Ax. 9.
(*g*) I. 5.
(*h*) I. 19.
(*i*) Constr. and I. 4.

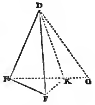
than EF (*h*); but EG is equal to BC (*i*), *therefore* BC *is greater than* EF.

SCHOLIA. 1. At (*e*) in the above demonstration it is *assumed* that "DH is less than DG," whereas it should have been proved, which may be done in the following manner. In the triangle DEH the external angle DHG is greater than the internal opposite angle DEG (*j*); and because in the triangle DEG the side DG is greater than DE (*k*), the angle DEG is greater than the angle DGE (*l*); therefore the angle DHG is greater than the angle DGE, *and the side* DG *greater than the side* DH (*m*).

(*j*) I. 16.
(*k*) Constr.
(*l*) I. 18.
(*m*) I. 19.

2. The foregoing proposition may also be readily demonstrated in the following manner.

CONSTRUCTION. *From the point* D *draw the straight line* DG, *making, with the side* DE, *which is not the greater, an angle* EDG *equal to the angle* A (*a*); *make* DG *equal to* AC (*b*), *and draw* EG. *Then bisect the angle* FDG (*c*) *with the straight line* DK, *cutting* EG *in* K, *and join* FK.



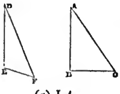
DEMONSTRATION. Because in the triangles DKG and DKF the sides DG and DF are equal (*d*), the side DK common to both, and the angle KDG equal to KDF (*d*), therefore the base KG is equal to KF (*e*); adding EK to each, we have EG equal to the sum of EK and KF (*f*), but EK and KF are together greater than EF (*g*), *therefore* EG *is greater than* EF.

(*a*) I. 23.
(*b*) I. 3.
(*c*) I. 9.
(*d*) Constr.
(*e*) I. 4.
(*f*) Ax. 2.
(*g*) I. 20.

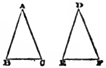
PROPOSITION XXV.

THEOREM.—*If two triangles* (ABC and DEF) *have two sides of the one respectively equal to two sides of the other* (BA and AC to ED and DF), *and if the third side* (BC) *of the one be greater than the third side* (EF) *of the other*, the angle (A) opposite to the greater side is greater than the angle (D) which is opposite to the less.

DEMONSTRATION. For the angle A is either equal to, less than, or greater than D. It is not equal, because then the base BC would be equal to EF (*a*), while by the hypothesis BC is greater than EF; and it is not less than D, because then the base BC would be less than EF (*b*); therefore *the angle* A *opposite to the greater side is greater than the angle* D *opposite to the less.*



(*a*) I. 4.
(*b*) I. 24.



SCHOLIUM. There is a similar mutual relation between propositions iv., viii., xxiv., and xxv., to that which has been pointed out in the scholium to proposition xix. This will be immediately obvious by placing these propositions in the form below.

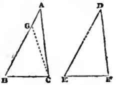
$$\left\{\begin{array}{ll}\text{Prop. IV.} & \left.\begin{array}{r}\textit{If } AB = DE,\\ AC = DF,\\ \textit{and } \angle A = \angle D,\end{array}\right\}\text{then BC} = \text{EF.}\\ \text{Prop. VIII.} & \left.\begin{array}{r}\textit{If } AB = DE,\\ AC = DF,\\ \textit{and } BC = EF,\end{array}\right\}\text{then } \angle\text{A} = \angle\text{D.}\end{array}\right.$$

$$\left\{\begin{array}{ll}\text{Prop. XXIV.} & \left.\begin{array}{r}\textit{If } AB = DE,\\ AC = DF,\\ \textit{and } \angle A > \angle D,\end{array}\right\}\text{then BC} > \text{EF.}\\ \text{Prop. XXV.} & \left.\begin{array}{r}\textit{If } AB = DE,\\ AC = DF,\\ \textit{and } BC > EF,\end{array}\right\}\text{then } \angle\text{A} > \angle\text{D.}\end{array}\right.$$

Here the propositions connected by a bracket are the *converse* of each other, the twenty-fourth and twenty-fifth propositions being the converse of each other in precisely the same way as the fourth and eighth are, as already explained in the scholium to the eighth proposition.

PROPOSITION XXVI.

THEOREM.—*If two triangles* (ABC and DEF) *have two angles of the one respectively equal to two angles of the other* (B to E and C to F), *and a side of the one equal to a side of the other, either* [1] *the sides adjacent to, or* [2] *the sides opposite to those equal angles* (BC to EF or BA to ED), the remaining angles and sides shall be respectively equal to one another.



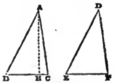
DEMONSTRATION. [1.] Let the equal sides be BC and EF, the sides adjacent to the equal angles, then the side BA is equal to ED.

For if it be possible, let one of them BA be the greater; *make* BG *equal to* ED (*a*), *and draw* GC. Then because in the triangles GBC and DEF the sides BG and BC are equal to the sides ED and EF (*b*), and the angle B to the angle E (*c*), therefore the angle GCB is equal to the angle F (*d*); but by the hypothesis the angle ACB is also equal to F, therefore the angle ACB is equal to GCB (*e*), the greater to the less, which is absurd; therefore *neither of the sides* BA *and* ED *is greater than the other, and therefore they are equal.* And because in the triangles ABC and DEF the sides BA and BC are equal to ED and EF, and

(*a*) I. 3.
(*b*) Constr. and Hypoth.
(*c*) Hypoth.
(*d*) I. 4.
(*e*) Ax. 1.
(*f*) I. 16.

the angle B to the angle E, therefore *the side* AC *is equal to* DF, *and the angle* A *to the angle* D (*d*).

[2.] Let the equal sides be BA and ED, opposite to the equal angles C and F, then the side BC is equal to EF.

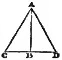


For if it be possible, let one of them BC be the greater; *make* BH *equal to* EF (*a*), *and join* AH. Then, because in the triangles ABH and DEF, the sides BA and BH are equal to ED and EF (*b*), and the angle B equal to the angle E (*c*), therefore the angle AHB is equal to the angle F (*d*); but the angle C is also equal to the angle F (*c*), therefore the angle AHB is equal to the angle C (*e*), that is, the external angle AHB of the triangle AHC is equal to the internal opposite angle C, which is impossible (*f*); therefore *neither of the sides* BC *and* EF *is greater than the other, and therefore they are equal.* And because in the triangles ABC and DEF the sides BA and BC are equal to ED and EF, and the angle B to the angle E, therefore *the side* AC *is equal to* DF, *and the angle* A *to the angle* D (*d*).

SCHOLIUM. It is evident that the triangles are themselves equal.

COROLLARY 1. *If a straight line* (AB) *be drawn from the vertex of an isosceles triangle* (ACD) *perpendicular to the base,* it will bisect the base, and also the angle (CAD) opposite to the base.

For in the triangles ABC and ABD the angles C and ABC are equal to the angles D and ABD (*a*), and the side AB, opposite the equal angles C and D, is common to both; therefore the angle CAB is equal to the angle DAB, and the side CB to BD (*b*); and therefore *the base* CD *is bisected, and also the angle* CAD *opposite to the base.*

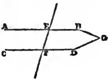


(*a*) Hypoth.
(*b*) I. 26.

COROLLARY 2. It is evident that a straight line which bisects the angle opposite to the base of an isosceles triangle, bisects also the base, and is perpendicular to it; and that a straight line drawn from the vertex, bisecting the base, is perpendicular to it, and bisects the opposite angle.

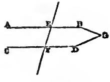
PROPOSITION XXVII.

THEOREM.—*If a straight line* (EF) *intersect two other straight lines* (AB and CD), *both in the same plane, and form alternate angles equal to each other* (AEF to EFD or BEF to EFC), these two straight lines shall be parallel.



DEMONSTRATION. For if possible let AB and CD not be parallel, but meet when produced on the side BD in some point, as G. Then, in the triangle EGF, the external angle AEF is greater than the in-

ternal opposite angle EFG (*a*); but it is also equal to it (*b*), which is absurd; therefore the lines AB and CD do not meet on the side BD, and in like manner it can be proved that they do not meet on the side AC. Since, then, *the lines* AB *and* CD, when produced on either side, do not meet, they *are parallel* (*c*).

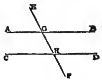


(*a*) I. 16.
(*b*) Hypoth.
(*c*) Def. 7.

SCHOLIUM. The condition that both the straight lines AB and CD shall be in the same plane is necessary to be introduced in the enunciation of this and the two following propositions; for it would be possible for two straight lines to accord with the remainder of the hypothesis, and yet not to be parallel if they were not in the same plane.

PROPOSITION XXVIII A.

THEOREM.—*If a straight line* (EF) *intersect two other straight lines* (AB and CD), *both in the same plane, and form an external angle equal to the internal and opposite angle upon the same side of the line* (EGB to GHD), the two straight lines shall be parallel.

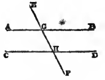


DEMONSTRATION. Because the angle EGB is equal to the angle GHD (*a*), and the angle EGB is also equal to the vertical angle AGH (*b*), therefore AGH is equal to GHD (*c*); but they are the alternate angles, *therefore* AB *is parallel to* CD (*d*).

(*a*) Hypoth.
(*b*) I. 15.
(*c*) Ax. 1.
(*d*) I. 27.

PROPOSITION XXVIII B.

THEOREM.—*If a straight line* (EF) *intersect two other straight lines* (AB and CD), *both in the same plane, and form internal angles at the same side* (BGH and GHD) *equal to two right angles*, the two straight lines shall be parallel.



(*a*) Hypoth.
(*b*) I. 13.
(*c*) Ax. 1.
(*d*) I. 27.

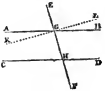
DEMONSTRATION. Because the angles BGH and GHD are equal to two right angles (*a*), and the angles AGH and BGH are also equal to two right angles (*b*), therefore the angles BGH and GHD are equal to the angles AGH and BGH (*c*); take away from both the common angle BGH, and

the remaining angles GHD and AGH are equal; but they are the alternate angles, *therefore* AB *is parallel to* CD (*d*).

SCHOLIUM. The twenty-eighth proposition of Euclid really consists of two distinct propositions, which are here distinguished and separately demonstrated.

PROPOSITION XXIX.

THEOREM.—*If a straight line* (EF) *intersect two parallel straight lines* (AB and CD), [1] it forms the alternate angles equal to one another (AGH to GHD), [2] and the external angle equal to the internal and opposite angle upon the same side (EGB to GHD), [3] and also the two internal angles on the same side (BGH and GHD), together equal to two right angles.



(*a*) I. 23.
(*b*) I. 27.
(*c*) Hypoth.
(*d*) Ax. 12.
(*e*) I. 15.
(*f*) Ax. 1.
(*g*) Ax. 2.
(*h*) I. 13.

DEMONSTRATION. [1.] For if AGH is not equal to GHD, let KG be drawn, making the angle KGH equal to the internal opposite angle GHD (*a*), and produce KG to L; then the line KL will be parallel to CD (*b*); but AB is also parallel to CD (*c*); therefore through the same point G two straight lines have been drawn parallel to the same straight line, which is impossible (*d*); *the alternate angles* AGH *and* GHD *are therefore equal.*

[2.] The external angle EGB is equal to the internal and opposite angle on the same side GHD; for the angle EGB is equal to the vertical angle AGH (*e*), and the angle AGH has been proved to be equal to the alternate angle GHD, therefore EGB is equal to GHD (*f*).

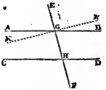
[3.] The two internal angles on the same side BGH and GHD are together equal to two right angles. For since EGB is equal to GHD, add BGH to both, therefore EGB and BGH are equal to GHD and BGH (*g*); but EGB and BGH are equal to two right angles (*h*), *therefore also* BGH *and* GHD *are equal to two right angles.*

SCHOLIA. 1. Euclid's twenty-ninth proposition really consists of three distinct propositions, which are here separated; [1] is the converse of proposition xxvii., [2] is the converse of proposition xxviii A, and [3] is the converse of proposition xxviii B.

2. The demonstration here given is similar to that of Playfair, and differs slightly from that given by Euclid, being proved by means of the axiom, "That through the same point two straight lines cannot be drawn parallel

to the same straight line," instead of being proved by the twelfth axiom as given by Simson, which, being the converse of another proposition (I. 17), requires itself to be demonstrated, which may be done in the following manner.

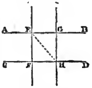
THEOREM. *If a straight line* (EF) *meets two straight lines* (AB and CD), *so as to make the two internal angles on the same side* (BGH and GHD) *together less than two right angles*, these straight lines (AB and CD) being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.



DEMONSTRATION. For if the two straight lines AB and CD do not meet when continually produced, they are parallel (*a*); and if GL be drawn, making the angles LGH and GHD together equal to two right angles, it will be parallel to CD (*b*). That is, through the point G the two straight lines GL and AB have been drawn both parallel to the line CD, which is impossible (*c*); therefore the *straight lines* AB *and* CD *are not parallel, but shall at length meet if continually produced.*

(*a*) Def. 7.
(*b*) I. 28 B
(*c*) Ax. 12.

COROLLARY. The parts of all perpendiculars to two parallel straight lines, intercepted between them, are equal.

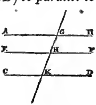


Let EF and GH be perpendicular to the two parallel straight lines AB and CD; and join EH. In the triangles FEH and GHE the angles FEH and GHE are equal, being alternate (*a*), and the angles EHF and HEG are equal, being alternate (*a*), and the side EH is common to both; therefore the remaining sides are respectively equal to each other (*b*), *the side* EF *to the side* GH.

(*a*) I. 29.
(*b*) I. 26.

PROPOSITION XXX.

THEOREM.—*If two straight lines* (AB and CD) *be parallel to the same straight line* (EF), they are parallel to each other.

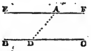


DEMONSTRATION. Let the straight line GHK intersect AB, EF, and CD. Then the angle AGK is equal to the alternate angle GHF (*a*), and the external angle GHF is equal to the internal opposite angle GKD (*a*), therefore the angle AGK is equal to the angle GKD (*b*); and because they are alternate angles, and are equal, therefore AB *is parallel to* CD (*c*).

(*a*) I. 29.
(*b*) Ax. 1.
(*c*) I. 27.

PROPOSITION XXXI.

PROBLEM.—Through a given *point* (A), to draw a straight line parallel to a given *straight line* (BC).



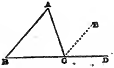
SOLUTION. *In the line* BC *take any point* D, *join* AD, *and at the point* A *form the angle* DAE *with the straight line* AD, *equal to the angle* ADC, *and on the opposite side of the line* AD (*a*), *and produce* EA *to* F; *then the line* EF *is parallel to* BC.

(*a*) I. 23.
(*b*) Solution.
(*c*) I. 27.

DEMONSTRATION. For the straight line AD, intersecting the lines EF and BC, forms the alternate angles DAE and ADC equal (*b*), therefore *the lines* EF *and* BC *are parallel* (*c*).

PROPOSITION XXXII A.

THEOREM.—*If any side* (BC) *of a triangle* (ABC) *be produced*, the external angle (ACD) is equal to the sum of the two internal and opposite angles (A and B).



CONSTRUCTION. *Through the point* C *draw* CE *parallel to the straight line* AB (*a*).

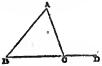
DEMONSTRATION. Because BA and CE are parallel (*b*), the angle ACE is equal to the alternate angle A (*c*), and the angle ECD is equal to the internal angle B (*c*), therefore *the whole external angle* ACD *is equal to the two internal angles* A *and* B (*d*).

(*a*) I. 31.
(*b*) Const.
(*c*) I. 29.
(*d*) Ax. 2.

SCHOLIUM. Euclid's thirty-second proposition really consists of two distinct theorems, which are here separately distinguished.

PROPOSITION XXXII B.

THEOREM.—*If any three angles are the internal angles of a triangle* (ABC), they are together equal to two right angles.



DEMONSTRATION. *Produce* BC *to* D. Then the angles ACB and ACD are equal to two right angles (*a*), they are also equal to the three internal angles A, B, and ACB (*b*); therefore *the three internal angles of the triangle* ABC *are together equal to two right angles* (*c*).

(*a*) I. 13.
(*b*) I. 32 A.
(*c*) Ax. 1.

COROLLARY 1. Any triangle can have but one right angle.

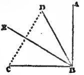
COROLLARY 2. In any triangle, if one angle be right, the other two are together equal to a right angle; and if one angle be equal to the other two, it is a right angle.

COROLLARY 3. If two triangles have two angles in the one respectively equal to two angles in the other, the remaining angles are also equal.

COROLLARY 4. In a right-angled isosceles triangle, each angle at the base is half a right angle.

COROLLARY 5. Each angle of an equilateral triangle is equal to a third part of two right angles, or to two-thirds of one right angle.

COROLLARY 6. From the foregoing corollary may be derived a method of trisecting a right angle.

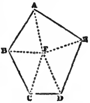


Upon any portion of the side CB construct an equilateral triangle CDB (*a*), and bisect the angle CBD by the line EB (*b*); then is the right angle ABC divided into three equal parts; for the whole angle CBD being equal to two-thirds of ABC (*c*), its halves are each equal to one-third, and the angle ABD is the remaining third part.

By successive bisections of the angles ABD, DBE, and EBC, the right angle ABC may be divided into 6, 12, 24, &c., equal parts.

(*a*) I. 1.
(*b*) I. 9.
(*c*) I. 32 B, cor. 6.

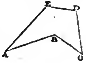
COROLLARY 7. All the internal angles of any rectilineal figure (ABCDE), together with four right angles, are equal to twice as many right angles as the figure has sides.



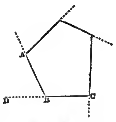
DEMONSTRATION. Take any point F within the figure, and draw the straight lines FA, FB, FC, FD, FE. There are formed as many triangles as the figure has sides, therefore all their angles taken together are equal to twice as many right angles as the figure has sides (*a*); but the angles at the point F are together equal to four right angles (*b*), therefore *all the internal angles of the figure* ABCDE, *together with four right angles, are equal to twice as many angles as the figure has sides.*

(*a*) I. 32 B.
(*b*) I. 13, cor. 3.

SCHOLIA. 1. It should be observed, that if the figure has a re-entrant angle, as ABC, the *internal* angle (although greater than two right angles) must be taken, and not the *external* angle.

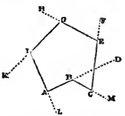


COROLLARY 8. The external angles of any rectilineal figure (ABC) are together equal to four right angles.



DEMONSTRATION. For each external angle ABD, with the adjacent internal angle ABC, is equal to two right angles (*a*); therefore all the external angles, together with all the internal angles, are equal to twice as many right angles as the figure has sides; but the internal angles, together with four right angles, are equal to twice as many right angles as the figure has sides (*b*); take away from both the internal angles, and *the external angles remain, equal to four right angles* (*c*).

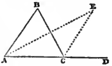
(*a*) I. 13.
(*b*) I. 32 B, cor. 7.
(*c*) Ax. 3.



SCHOLIA. 2. If the figure has a *re-entrant* angle, as ABC, the angle DBC, or the difference between the angle ABC and two right angles, must be deducted from the sum of all the exterior angles. Thus the sum of the angles LAB, MCE, FEG, HGI, KIA, less the angle DBC, will equal four right angles.

3. The following theorem is introduced as depending upon the preceding, and because of its use in explaining the principles of construction of the quadrant and sextant.

THEOREM. *If an exterior angle* (BCD) *of a triangle* (ABC) *be bisected, and also one of the interior and opposite angles* (BAC), the angle (E) formed by the bisecting lines is equal to half the other interior and opposite angle (B) of the triangle.



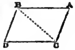
DEMONSTRATION. The lines AE and CE will meet; for because the angle BAC is less than the exterior angle BCD (*a*), the half of the former EAC is less than the half of the latter ECD; add ECA to both, and EAC and ECA are less than ECD and ECA; but ECD and ECA are together equal to two right angles, therefore EAC and ECA are together less than two right angles, and therefore the lines AE and CE must meet on the same side of AC as those angles (*b*); let them meet in E. Then, because ECD is the exterior angle of the triangle EAC, it equals the two interior angles EAC and E (*c*), and therefore twice ECD equals twice the angles EAC and E (*d*), that is, the angle BCD is equal to the sum of the angle BAC and twice the angle E; but the angle BCD is equal to the two interior angles BAC and B (*c*), therefore the sum of BAC and twice E is equal to the sum of BAC and B (*e*), and taking BAC from both, therefore *the angle* B *is equal to twice the angle* E (*f*).

(*a*) I. 16.
(*b*) Theor. attached to I. 29.
(*c*) I. 32 A.
(*d*) Ax. 6.
(*e*) Ax. 1.
(*f*) Ax. 3.

PROPOSITION XXXIII.

THEOREM.—*If straight lines* (AC and BD) *join the adjacent extremities of two equal and parallel straight lines* (AB and CD), they are themselves equal and parallel.

CONSTRUCTION. *Draw the diagonal* BC.

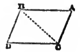


DEMONSTRATION. Because in the triangles ABC and DBC the sides AB and CD are equal (*a*), the side BC common to both, and the angle ABC equal to the alternate angle BCD (*b*), therefore *the two sides* AC *and* BD *are equal* (*c*), and also the two angles ACB and CBD are equal (*c*); and because the straight line BC intersects the straight lines AC and BD, and forms alternate angles ACB and CBD equal to each other, therefore AC *is parallel to* BD (*d*).

(*a*) Hypoth.
(*b*) I. 29.
(*c*) I. 4.
(*d*) I. 27.

PROPOSITION XXXIV.

THEOREM.—*If a figure* (ACDB) *be a parallelogram,* [1] the opposite sides are equal to one another (AB to CD and AC to BD), [2] as are also the opposite angles (A to D and ACD to DBA), [3] and the parallelogram is bisected by its diagonal (CB).



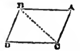
(a) Def. 26.
(b) I. 29.
(c) I. 26.
(d) I. 26, schol.
(e) Ax. 2.

DEMONSTRATION. Because AB is parallel to CD (*a*), the angle ABC is equal to the alternate angle BCD (*b*); and because AC is parallel to BD (*a*), the angle ACB is equal to the alternate angle DBC (*b*). Then, because in the two triangles ABC and DBC the two angles ABC and ACB are equal to BCD and DBC, and the side BC common to both, [1] therefore *the sides* AB *and* AC *are equal to the opposite sides* CD *and* BD (*c*), the angle A is equal to the angle D (*c*), and the triangle ABC is equal to the triangle DBC (*d*), [3] therefore *the parallelogram is bisected by the diagonal* BC.

[2.] Also because the angle ABC is equal to BCD, and the angle ACB is equal to DBC, therefore *the whole angle* ACD *is equal to the opposite whole angle* ABD (*e*), *and the angle* A *has been proved to be equal to the opposite angle* D.

SCHOLIUM. This theorem consists of three distinct propositions, which are here separately distinguished by numbers. The converse propositions to the first and second may be demonstrated as follows.

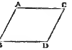
THEOREM. *If in any four-sided figure* (ABDC) *the opposite sides are equal,* it is a parallelogram.



(a) Hypoth.
(b) I. 8.
(c) I. 27.
(d) Def. 26.

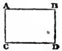
DEMONSTRATION. Draw BC. Then, because in the triangles ABC and DBC the sides AC and AB are respectively equal to the sides BD and CD (*a*), and BC is common to both, therefore the angle ABC is equal to the angle BCD, and the angle ACB to the angle CBD (*b*). And because the alternate angles ABC and BCD are equal, therefore the straight lines AB and CD are parallel; also the alternate angles ACB and CBD being equal, the straight lines AC and BD are parallel (*c*), and therefore *the figure* ABDC *is a parallelogram* (*d*).

THEOREM. *If in any four-sided figure* (ABDC) *the opposite angles are equal,* it is a parallelogram.



(a) Hypoth.
(b) Ax. 2.
(c) I. 32 B, cor. 7
(d) I. 28 B.
(e) Def. 26.

DEMONSTRATION. Because the angle A is equal to D, and the angle B to C (*a*), therefore A and B are equal to D and C (*b*), and the four angles are together double of A and B; but the four angles together are equal to four right angles (*c*), therefore A and B are equal to two right angles, and the line AC is parallel to BD (*d*); and in a similar manner it may be proved that AB is parallel to CD, *therefore the figure* ABDC *is a parallelogram* (*e*).



Corollary 1. *If a parallelogram* (ABDC) *have one right angle*, all its angles are right angles.

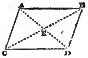
For since the adjacent angles A and B are together equal to two right angles (*a*), if one of them is a right angle (*b*), *the other must also be a right angle; and the angles* C *and* D *are right angles, being equal to their opposite angles* B *and* A (*c*).

(*a*) I. 29.
(*b*) Hypoth.
(*c*) I. 34.

Corollary 2. *If two parallelograms have an angle of the one equal to an angle of the other*, the remaining angles shall be respectively equal.

For the angles opposite the equal angles are equal to them (*a*), and therefore to each other (*b*); and the angles adjacent to the equal angles are equal, being, with these equals, equal to two right angles (*c*).

(*a*) I. 34.
(*b*) Ax. 1.
(*c*) I. 29.

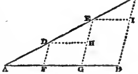


Corollary 3. The diagonals of a parallelogram bisect each other.

For since, in the triangles AEC and DEB, the side AC is equal to the side BD (*a*), the angle EAC to EDB, and the angle ECA to EBD (*b*), therefore *the side* AE *is equal to* ED, *and the side* CE *to* EB (*c*).

(*a*) Hypoth.
(*b*) I. 29.
(*c*) I. 26.

Scholium. By means of the thirty-fourth proposition we may divide a finite straight line into any given number of equal parts in the following manner.

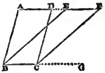


Let AB be the given line. Draw AC, making any angle with AB; take any part AD, and make DE, EC equal to it until as many equal parts have been taken as that into which the line AB is to be divided. Join CB, and draw EG, DF parallel to CB, then the line AB shall be divided into the required number of equal parts.

Draw DH and EI parallel to AB; then they are also parallel to each other (*a*); therefore the angles CEI, EDH, and DAF are equal to each other (*b*), and also the angles C, DEH, and ADF are equal to each other (*b*), and the sides CE, ED, and DA being equal (*c*), the sides EI, DH, and AF are also equal (*d*). Then, because FG and GB are equal to DH and EI (*e*), *therefore they are equal to* AF (*f*).

(*a*) I. 30.
(*b*) I. 29.
(*c*) Hypoth.
(*d*) I. 26.
(*e*) I. 34.
(*f*) Ax. 1.

PROPOSITION XXXV.



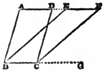
Theorem.—*If parallelograms* (ABCD and EBCF) *are upon the same base and between the same parallels,* they are equal in area.

Construction. *Produce the side* BC *to* G.

Demonstration. Because the lines AB and DC are parallel (*a*), the angle DCG is equal to the internal angle ABG (*b*); and

(*a*) Def. 26.
(*b*) I. 29.

because EB and FC are parallel (*a*), the angle FCG is equal to the internal angle EBG (*b*); therefore the angle DCF is equal to the angle ABE (*c*). Then, because in the triangles DCF and ABE the two sides DC and CF are equal to the two sides AB and BE (*d*), and the angle DCF is equal to ABE, therefore the triangles themselves are equal (*e*). Then, from the quadrilateral ABCF take away the triangle DCF, the remainder is the parallelogram ABCD; and from the same quadrilateral take away the triangle ABE, the remainder is the parallelogram EBCF; therefore *the parallelograms* ABCD *and* EBCF *are equal in area* (*c*).



(*a*) Def. 26.
(*b*) I. 29.
(*c*) Ax. 3.
(*d*) I. 34.
(*e*) I. 4.

SCHOLIA. 1. The demonstration of Euclid has not been adhered to in this proposition, in order to avoid its division into distinct cases, according to the relative positions of the lines AD and EF.

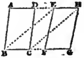
2. The word "equal" is, in this and several following propositions, used by Euclid in a sense somewhat different to that in which it has been employed in the previous portion of the Elements; namely, to denote simply equality in area, and not absolute sameness in form. In order to avoid any ambiguity, the phrase "equal in area" has been substituted for equal; and the latter is only used to signify perfect identity of form.

PROPOSITION XXXVI.

THEOREM.—*If parallelograms* (ABCD and EFGH) *are upon equal bases and between the same parallels,* they are equal to one another in area.

CONSTRUCTION. *Draw* BE *and* CH.

DEMONSTRATION. Because the lines BC and EH are equal to the same FG (*a*), they are equal to one another (*b*); but they are also parallel (*c*); therefore BE and CH, which join their extremities, are parallel (*d*), and EBCH is a parallelogram equal in area both to ABCD and EFGH (*e*); and therefore *the parallelograms* ABCD *and* EFGH *are equal in area* (*b*).

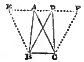


(*a*) Hypoth. and I. 34.
(*b*) Ax. 1.
(*c*) Hypoth.
(*d*) I. 33.
(*e*) I. 35.

PROPOSITION XXXVII.

THEOREM.—*If triangles* (ABC and DBC) *are upon the same base and between the same parallels,* they are equal to one another in area.

CONSTRUCTION. *Produce* AD *both ways to the points* E *and* F; *through* B *draw* BE *parallel to* CA (*a*), *and through* C *draw* CF *parallel to* BD (*a*).



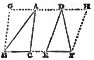
(*a*) I. 31
(*b*) I. 35.
(*c*) I. 34.
(*d*) Ax. 7.

DEMONSTRATION. Because the parallelograms EBCA and DBCF are upon the same base and between the same parallels, they are equal in area (*b*); and because the diagonals BA and CD bisect the equal parallelograms (*c*), the triangle ABC is half the parallelogram EBCA, and the triangle DBC is half the parallelogram DBCF; therefore *the triangle* ABC *is equal in area to the triangle* DBC (*d*).

PROPOSITION XXXVIII.

THEOREM.—*If triangles* (ABC and DEF) *are upon equal bases and between the same parallels,* they are equal to one another in area.

CONSTRUCTION. *Produce* AD *both ways to the points* G *and* H; *through* B *draw* BG *parallel to* CA (*a*), *and through* F *draw* FH *parallel to* ED (*a*).



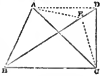
(*a*) I. 31.
(*b*) I. 36.
(*c*) I. 34.
(*d*) Ax. 7.

DEMONSTRATION. Because the parallelograms GBCA and DEFH are upon equal bases and between the same parallels, they are equal in area (*b*); and because the diagonals BA and FD bisect the equal parallelograms (*c*), the triangle ABC is half the parallelogram GBCA, and the triangle DEF is half the parallelogram DEFH; therefore *the triangle* ABC *is equal to the triangle* DEF (*d*).

COROLLARY. Hence a straight line drawn from the vertex of a triangle bisecting its base, also bisects the triangle.

PROPOSITION XXXIX.

THEOREM.—*If triangles equal in area* (ABC and DBC) *be upon the same base, and upon the same side of it,* they are between the same parallels.



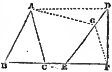
DEMONSTRATION. Draw AD; then it is parallel to BC; for if not, through the point A draw AE parallel to BC (*a*), and join EC. Then the triangles ABC and EBC are equal because they are upon the same base (*b*) and between the same parallels (*c*); but ABC is also equal to DBC (*b*); therefore DBC is equal to EBC (*d*), the greater to the less; which is absurd. Therefore AE is not parallel to BC; and in the same manner it may be proved that no other line than AD is parallel to it; therefore AD *is parallel to* BC.

(*a*) I. 31.
(*b*) Hypoth.
(*c*) I. 37.
(*d*) Ax. 1.

SCHOLIUM. This proposition is the converse of the thirty-seventh, and is demonstrated by the "reductio ad absurdum."

PROPOSITION XL.

THEOREM.—*If triangles equal in area* (ABC and DEF) *are upon equal bases in the same straight line, and on the same side of it,* they are between the same parallels.



DEMONSTRATION. Draw AD; then it is parallel to BF; for if not, through the point A draw AG parallel to BF (*a*), and join GF. Then the triangles ABC and GEF are equal, because they are on equal bases (*b*), and between the same parallels (*c*); but ABC is also equal to DEF (*b*), therefore DEF is equal to GEF (*d*), the greater to the less; which is absurd. Therefore AG is not parallel to BF; and in the same manner it may be proved that no other line than AD is parallel to it; therefore AD *is parallel to* BF.

(*a*) I. 31.
(*b*) Hypoth.
(*c*) I. 38.
(*d*) Ax. 1.

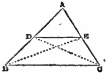
SCHOLIUM. This proposition is the converse of the thirty-eighth, and is demonstrated by the "reductio ad absurdum."

The following corollaries may be drawn from this and the preceding proposition.

COROLLARY 1. Any parallelogram or triangle is equal in area to a right-angled parallelogram or triangle having an equal base and altitude.

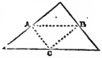
COROLLARY 2. A straight line DE bisecting the two sides of a triangle is parallel to its base.

For join BE and CD; then, because both the lines BE and CD bisect the triangle ABC (*a*), therefore the triangles DBC and EBC are equal in area; but they are also on the same base, and therefore between the same *parallels* DE *and* BC (*b*).



(*a*) I. 38, cor.
(*b*) I. 39.

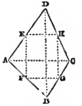
COROLLARY 3. Straight lines joining the points of bisection ABC of the three sides of a triangle, divide it into four equal triangles.



COROLLARY 4. The straight lines joining the points of bisection of the three sides of a triangle are respectively equal to half the parallel sides.

COROLLARY 5. If the sides of a quadrilateral figure ABCD be bisected, and the points of bisection of each pair of conterminous sides joined by straight lines, those lines will form a parallelogram EFGH whose area is equal to half that of the quadrilateral.

Draw AC and BD. The lines EH and FG are equal; being each equal to half AC (*a*), they are also parallel (*b*); therefore EFGH *is a parallelogram* (*c*). Further, the triangle DEH is equal in area to one-fourth of DAC, and BFG is equal in area to one-fourth of BAC (*d*); therefore the two triangles DEH and BFG are together equal in area to one-fourth of the whole figure ABCD; in like manner the two triangles AEF and CHG may be shown to be together equal to one-fourth of the whole; therefore DEH, AEF, BFG, and CHG are together equal in area to one-half the whole figure ABCD, *and so the parallelogram* EFGH *must also be equal in area to half the same.*



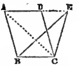
(*a*) Ax. 7, and I. 40, cor. 4.
(*b*) I. 30, and I. 40, cor. 2.
(*c*) I. 33.
(*d*) I. 40, cor. 3.

PROPOSITION XLI.

THEOREM.—*If a parallelogram* (ABCD) *and a triangle* (EBC) *be upon the same base, and between the same parallels,* the parallelogram is double of the triangle.

CONSTRUCTION. *Draw the diagonal* AC.

DEMONSTRATION. Then, because the two triangles ABC and EBC are on the same base and between the same parallels (*a*), therefore they are equal (*b*). But the parallelogram ABCD is double of the triangle ABC (*c*), therefore ABCD is also double of EBC.

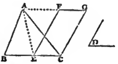


(*a*) Hypoth.
(*b*) I. 37.
(*c*) I. 34.

COROLLARY. Hence, if a parallelogram and a triangle have equal bases and are between the same parallels, the parallelogram is double of the triangle.

PROPOSITION XLII.

PROBLEM.—To construct a parallelogram equal in area to a given *triangle* (ABC) and having an angle equal to a given *rectilineal angle* (D).



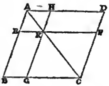
SOLUTION. *Bisect* BC *in* E (*a*), *join* AE, *and at the point* E *make the angle* FEC *equal to the given angle* D (*b*); *also, through* A *draw* AG *parallel to* BC (*c*), *and through* C *draw* CG *parallel to* EF (*c*). FECG *will be the parallelogram required.*

(*a*) I. 10.
(*b*) I. 23.
(*c*) I. 31.
(*d*) Const.
(*e*) I. 38.
(*f*) I. 41.
(*g*) Ax. 6.

DEMONSTRATION. Because the triangles ABE and AEC are upon equal bases BE and EC, and between the same parallels BC and AG (*d*), therefore they are equal in area (*e*); and the triangle ABC is double of the triangle ABE; but, because the parallelogram FECG and the triangle AEC are upon the same base EC and between the same parallels BC and AG (*d*), therefore the parallelogram is double of the triangle (*f*); *and the parallelogram* FECG *is equal in area to the triangle* ABC (*g*), *and has one of its angles equal to the given angle* D.

PROPOSITION XLIII.

THEOREM.—*The complements* (BK and KD) *of the parallelograms* (EH and GF), *which are about the diagonal of any parallelogram* (ABCD), are equal in area to one another.



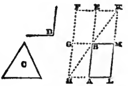
DEMONSTRATION. Because the diagonal of a parallelogram bisects it (*a*), the triangle ABC is equal to the triangle ADC, and the triangles AEK and KGC to the triangles AHK and KFC; then, if from the equals ABC and ADC the equals AEK and AHK, and also the equals KGC and KFC, be taken away, *the remaining complements,* BK *and* KD, *will be equal in area* (*b*).

(*a*) I. 34.
(*b*) Ax. 3.

COROLLARY. The parallelograms about the diagonal, and also their complements, are equiangular with the original parallelogram.

PROPOSITION XLIV.

Problem.—Upon a given *finite straight line* (AB) to construct a parallelogram which shall be equal in area to a given *triangle* (C), and have one of its angles equal to a given *rectilineal angle* (D).



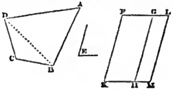
(a) I. 42.
(b) I. 31.
(c) I. 29.
(d) Theor., I. 29.
(e) I. 43.
(f) Constr.
(g) Ax. 1.
(h) I. 15.

Solution. *Produce* AB *to* E, *and upon* BE *construct a parallelogram* BEFG *equal in area to the triangle* C, *and having the angle* EBG *equal to the given angle* D (*a*). *Produce* FG *to* H, *through* A *draw* AH *parallel to* GB (*b*), *and join* HB. Then, because the straight line FH falls upon the two parallel lines FE and HA, the angles F and AHF are together equal to two right angles (*c*); and therefore the angles F and BHF are together less than two right angles; but the angles F and BHF are the interior angles made by HF with FE and HB on the same side; wherefore, if the straight lines FE and HB be produced, they shall meet (*d*). *Let them meet in* K, *through* K *draw* KL *parallel to* EA (*b*), *and produce* GB *and* HA *to meet* KL *in the points* M *and* L; *then* BALM *is the parallelogram required.*

Demonstration. Because FHLK is a parallelogram, of which HK is the diagonal, GA and EM the parallelograms about that diagonal, and FB and BL their complements, therefore BL is equal to FB (*e*); but FB is equal to the triangle C (*f*), therefore BL is equal to the triangle C (*g*). And because the angle GBE is equal to the angle ABM (*h*), and also to the angle D (*f*), the angle ABM is equal to the angle D (*g*). Therefore *the parallelogram* BL *constructed upon the given line* AB *is equal in area to the given triangle* C, *and has an angle equal to the given angle* D.

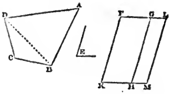
PROPOSITION XLV.

Problem.—To construct a parallelogram equal in area to a given *rectilineal figure* (ABCD) and having an angle equal to a given *rectilineal angle* (E).



Solution. *From any angle* D *draw a line* DB *dividing the given figure into triangles. Then construct a parallelogram* FH *equal in area to the triangle* ABD,

and having the angle K *equal to the given angle* E (*a*); *and upon the straight line* GH *construct a parallelogram* GM *equal in area to the triangle* DBC, *and having the angle* GHM *equal to the given angle* E (*b*); *then* FKML *will be the required parallelogram.*



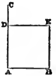
(*a*) I. 42.
(*b*) I. 44.
(*c*) I. 29.
(*d*) Constr.
(*e*) Ax. 1.
(*f*) I. 14.
(*g*) I. 30.
(*h*) Ax. 2.

Demonstration. Because the straight lines FK and GH are parallel, therefore the internal angles K and GHK are together equal to two right angles (*c*); but the angles K and GHM being both equal to the given angle E (*d*), are equal to one another (*e*); therefore the angles GHK and GHM are together equal to two right angles (*e*), and therefore KH and HM are in the same straight line (*f*). Because the straight line GH intersects the parallels FG and KM, the alternate angles FGH and GHM are equal (*c*); but the internal angles GHM and HGL are together equal to two right angles (*c*), therefore the angles FGH and HGL are together equal to two right angles (*e*), and therefore FG and GL are in the same straight line (*f*). Then, because FK and LM are both parallel to GH, therefore they are parallel to each other (*g*); and FL being parallel to KM, FKML is a parallelogram. And because the parallelogram FH is equal in area to the triangle ABD, and the parallelogram GM to the triangle DBC, therefore *the whole parallelogram* FKML *is equal in area to the whole figure* ABCD (*h*), *and has the angle* K *equal to the given angle* E.

Corollary. By means of this proposition, and that immediately preceding it, a parallelogram can be constructed *on a given line* equal in area to a given rectilineal figure, and having an angle equal to a given rectilineal angle, by constructing on the given line a parallelogram equal in area to the first triangle ABD.

PROPOSITION XLVI.

Problem.—Upon a given *finite straight line* (AB) to construct a square.

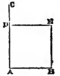


Solution. *From the point* A *draw* AC *perpendicular to* AB (*a*), *and make* AD *equal to* AB (*b*); *through the point* D *draw* DE *parallel to* AB (*c*), *and through the point* B *draw* BE *parallel to* AD (*c*); *then* DABE *is the required square.*

(*a*) I. 11.
(*b*) I. 3.
(*c*) I. 31.
(*d*) Constr

Demonstration. The side DE being parallel to the opposite side AB, and the side BE to AD (*d*), the

figure DABE is a parallelogram (e); and therefore its opposite sides are equal (f), that is DE to AB, and BE to AD; and because AD is equal to AB (d), therefore all the sides are equal (g); but the angle A is a right angle, wherefore *the four-sided figure* DABE *has all its sides equal and one of its angles a right angle; therefore it is a square* (h), *and it is constructed on the line* AB.

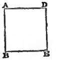


(d) Constr.
(e) Def. 26.
(f) I. 34.
(g) Ax. 1.
(h) Def. 28.

Scholium. The definition of a square, as given by Euclid, viz. "a four-sided figure, which has all its sides equal, *and all its angles right angles*," is more than sufficient, and really involves a theorem. It is only necessary to state in the definition that *one* of its angles is a right angle, and the proposition that its remaining angles are right may be demonstrated as follows.

Theorem. *If a four-sided figure* (DABE) *be a square*, all its angles are right angles.

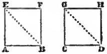
Demonstration. The opposite sides AB and DE are parallel (a), and they are met by the line AD, therefore the angles A and D are together equal to two right angles (b); but A is a right angle (a), therefore D is a right angle. And because, in the parallelogram DABE, the angles E and B are respectively opposite to A and D, which are right angles, therefore E and B are right angles; *and therefore all the angles of the square* DABE *are right angles.*



(a) Constr.
(b) I. 29.

Corollary 1. *If two squares are constructed on equal straight lines* AB and CD, they are equal.

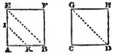
Demonstration. Draw the diagonals EB and GD. Because, in the triangles EAB and GCD, the sides EA and AB are respectively equal to GC and CD (a), and the angle A to the angle C, therefore the triangles are equal (b). And because the squares AF and CH are doubles of the triangles EAB and GCD (c), *therefore they are equal* (d).



(a) Hypoth. and Def.
(b) I. 4.
(c) I. 34.
(d) Ax. 6.

Corollary 2. *If two squares* AF *and* CH *are equal*, their sides are equal.

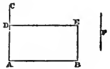
Demonstration. For, if it be possible, let one of them AF be the greater; take AK equal to CD, and AI equal to CG (a), and join IK. Then the triangles IAK and GCD are equal (b); but the triangles EAB and GCD are equal, being halves of the equal squares AF and CH (c), therefore the triangle IAK is equal to EAB (d), a part to the whole, which is absurd; *therefore neither of the sides* AB *or* CD *is greater than the other, but they are equal.*



(a) I. 3.
(b) I. 4.
(c) Ax. 7.
(d) Ax. 1.

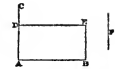
Corollary 3. To construct a rectangle under two given *finite straight lines* F and AB.

Solution. *From the point* A *draw* AC *perpendicular to* AB (a), *and make* AD *equal to* F (b); *through the point* D *draw* DE *parallel to* AB (c), *and through the point* B *draw* BE *parallel to* AD (c); *then* DABE *is the rectangle required.*



(a) I. 11.
(b) I. 3.
(c) I. 31.

DEMONSTRATION. The side DE being parallel to the opposite side AB, and the side BE to AD (*d*), the figure DABE is a parallelogram (*e*); and the angle A is a right angle (*d*), *therefore it is a rectangle* (*f*), *and it is constructed under the two straight lines* F *and* AB.

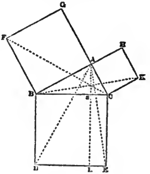


(*d*) Constr.
(*e*) Def. 26.
(*f*) Def. 27.

PROPOSITION XLVII.

THEOREM.—*If a triangle* (ABC) *be right-angled*, the square which is constructed upon the side (BC) subtending the right angle is equal in area to the sum of the squares constructed upon the sides (AB and AC) which form the right angle.

CONSTRUCTION. *On the sides* AB, BC, *and* AC, *construct the squares* BG, BE, *and* CH (*a*); *through* A *draw* AL *parallel to* BD (*b*), *and join* AD *and* FC.



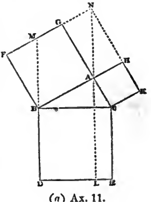
DEMONSTRATION. Because the angles FBA and CBD are both right angles (*a*), therefore they are equal (*c*); add to both the angle ABC, and the angle FBC is equal to ABD (*d*). Because the sides FB and BC are respectively equal to AB and BD (*e*), and the angle FBC to the angle ABD, therefore the triangle FBC is equal to the triangle ABD (*f*). Because the angles GAB and BAC are both right angles, therefore GA and AC are in the same straight line (*g*). Now the parallelogram BL is double of the triangle ABD, because they are on the same base BD and between the same parallels BD and AL (*h*); and the square GB is double of the triangle FBC, being on the same base FB and between the same parallels FB and GC (*h*). But the doubles of equals are equal to one another (*i*), and *therefore the parallelogram* BL *is equal in area to the square* GB. And in the same manner, by joining AE and BK, it may be proved that *the parallelogram* CL *is equal in area to the square* CH. *Therefore the whole square* BDEC *is equal in area to the two squares* BG *and* CH.

(*a*) I. 46.
(*b*) I. 31.
(*c*) Ax. 11.
(*d*) Ax. 2.
(*e*) Def. 28.
(*f*) I. 4.
(*g*) I. 14.
(*h*) I. 41.
(*i*) Ax. 6.

SCHOLIA. 1. It should be remarked that the line AL is a perpendicular from the point A on to the hypotenuse BC, dividing it into two segments B*e* and *e*C, and that the square described on either side of the triangle ABC is equal in area to the rectangle under the whole hypotenuse and the segment of the hypotenuse adjacent to the same side.

2. This proposition may be demonstrated in a variety of ways, but that adopted by Euclid has the preference of the others. The following mode of proving it is given because it is capable of being generalised, as is done in the third scholium.

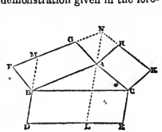
CONSTRUCTION. *Upon the sides* AB, AC, *and* BC, *construct the squares* BG, CH, *and* BE; *produce* FG *and* KH *to meet in* N; *draw* NA, *and produce it to meet* DE *in* L; *and produce* DB *to meet* FG *in* M.



DEMONSTRATION. The angles MBC and FBA, being both right angles, are equal (*a*); take from each the angle MBA, and the remaining angles FBM and ABC are equal (*b*); also the angles F and CAB are equal, being both right angles (*a*). Then, because in the triangles MFB and CAB the angles F and FBM are equal to the angles CAB and ABC, and the side FB is equal to AB, therefore the remaining sides are equal (*c*), FM to AC and MB to CB. Then, because FN is parallel to BH, and GC to NK, therefore GH is a parallelogram, and GN is equal to AH or AC (*d*); but FM is equal to AC, therefore GN is equal to FM; add to each the MG, and FG is equal to MN. But FG equal BA, therefore MN equal BA, and they are parallel; therefore the straight lines BM and AN joining their extremities are equal and parallel (*e*), and MBAN is a parallelogram. Then, because the parallelogram MBAN and the square BG are upon the same base AB and between the same parallels FN and BA, therefore they are equal in area (*f*). Also, because the parallelograms MBAN and BL are upon equal bases MB and BD, and between the same parallels MD and NL, they are equal in area (*g*); therefore the parallelogram BL is equal to the square BG (*h*); and in like manner it may be proved that the parallelogram CL is equal in area to the square CH, *and therefore that the whole square* BE *is equal in area to the squares* BG *and* CH.

(*a*) Ax. 11.
(*b*) Ax. 3.
(*c*) I. 26.
(*d*) I. 34.
(*e*) I. 33.
(*f*) I. 35.
(*g*) I. 36.
(*h*) Ax. 1.

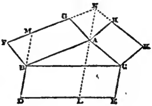
3. The forty-seventh proposition is only a particular case of the following theorem, as will be seen by comparing the demonstration given in the foregoing scholium with that given below.



THEOREM. *If parallelograms* (FBAG and HACK) *be constructed upon two of the sides* (AB and AC) *of any triangle* (ABC), *and their sides* (FG and KH) *parallel to the sides of the triangle be produced to meet in a point* (N); *if a straight line* (NA) *be drawn from that point to the vertex of the triangle, and if a parallelogram* (BDEC) *be constructed upon the base of the triangle whose other sides are equal*

and parallel to that straight line, then the last parallelogram (BDEC) is equal in area to the two former (FBAG and HACK).

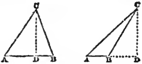
CONSTRUCTION. *Produce* DB *to meet* FG *in* M, *and* NA *to meet* DE *in* L.



DEMONSTRATION. The parallelograms FA and BN are between the same parallels FN and BA and upon the same base AB, therefore FA is equal in area to BN (*a*); and because the parallelograms BN and BL are between the same parallels DM and LN, and upon equal bases BD and NA, therefore BN is equal in area to BL (*b*); and therefore the parallelogram FA is equal in area to BL (*c*); and in the same manner it may be proved that HC is equal in area to CL, *therefore the whole parallelogram* BDEC *is equal in area to the parallelograms* FA *and* HC.

(*a*) I. 35.
(*b*) I. 36.
(*c*) Ax. 1.

COROLLARY 1; THEOREM. *If a perpendicular* (CD) *be drawn from the vertex of a triangle* (ABC) *cutting the base*, the difference of the squares on the sides (AC and CB) is equal to the difference of the squares on the segments of the base (AD and DB).

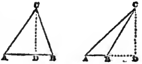


DEMONSTRATION. For the square on AC is equal in area to the squares on AD and CD (*a*), and the square on CB is equal in area to the squares on DB and CD (*a*); therefore the difference of the squares on AC and CB is equal to the difference of the sum of the squares on AD and CD, and the sum of the squares on DB and CD (*b*); or, taking from each the common square on CD, *equal to the difference of the square on* AD *and the square on* DB.

(*a*) I. 47.
(*b*) Ax. 3.

SCHOLIUM. This theorem applies whether the base is cut *internally*, as in the left-hand figure, or the base produced is cut *externally*, as in that on the right hand. See Scholium to Definition 2.

COROLLARY 2; THEOREM. *If a perpendicular* (CD) *be drawn from the vertex of a triangle* (ABC) *cutting the base*, the sum of the squares on one side and the alternate segment (AC and DB) is equal to the sum of the squares on the other side and the alternate segment (BC and AD).

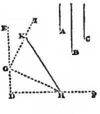


DEMONSTRATION. For the square on AC is equal in area to the sum of the squares on AD and CD (*a*), and the sum of the squares on DB and CD is equal in area to the square on CB (*a*); adding these equals together, we have the sum of the squares on AC, DB, and CD, equal to the sum of the squares on CB, AD, and CD (*b*); or, taking away the common square on CD, *the sum of the squares on* AC *and* DB *is equal to the sum of the squares on* CB *and* AD

(*a*) I. 47.
(*b*) Ax. 2.

Corollary 3; Problem. To construct a square equal in area to the sum of *two or more given squares.*

Solution. *Let the straight lines* A, B, *and* C *be sides of the given squares. Draw* ED *and* DF *at right angles, and make* GD *equal to* A *and* DH *equal to* B; *join* GH, *and draw* GI *perpendicular to it; make* GK *equal to* C, *and join* KH, *then a square constructed on the line* KH *shall be the square required.*

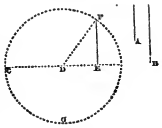


(*a*) I. 47.

Demonstration. For the square on KH is equal in area to the sum of the squares on GK or C, and on GH (*a*), but the square on GH is equal in area to the squares on GD and DH, or on A and B (*a*); *therefore the square on* KH *is equal in area to the sum of the squares on* A, B, and C.

Corollary 4; Problem. To construct a square equal in area to the difference of *two given squares.*

Solution. *Let* A *and* B *be sides of the given squares. Draw* CD *equal to the side of the greater square* B, *and produce it until the produced part* DE *is equal to the side of the other square* A. *From the center* D, *at the distance* CD, *describe the circle* CFG, *and through* E *draw* EF *perpendicular to* CE, *to meet the circle in* F; *then* FE *is a side of the required square.*

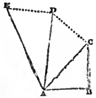


(*a*) I. 47.

Demonstration. Join DF. The square on DF, or its equal B, is equal in area to the sum of the squares on DE or A, and on EF (*a*); therefore *if the square on* DE *be taken from the square on* DF, *the difference is equal in area to the square on* EF.

Corollary 5; Problem. To find geometrical values of $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, &c.

Solution. *Take* AB *equal to* 1; *draw* CB *perpendicular to* AB, *and also equal to* 1; *join* AC, *and draw* DC *perpendicular to it and equal to* 1; *join* DA, *and draw* DE *perpendicular to it and equal to* 1, *&c.*; *then the lines* AC, AD, AE, &c., *are respectively equal to* $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, &c.



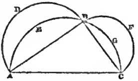
(*a*) I. 47.
(*b*) Const.

Demonstration. The square on AC equals the sum of the squares on AB and BC (*a*); the squares on AB and BC each equal 1 (*b*); therefore the square on AC equals 2, and AC equals $\sqrt{2}$. Again, the square on AD equals the sum of the squares on AC and DC (*a*); the square on AC equals 2, and the square on DC equals 1 (*b*); therefore the square on AD equals 3, and AD equals $\sqrt{3}$. *Therefore the square roots of the natural numbers* 1, 2, 3, 4, &c., *are represented geometrically by the lines* AB, AC, AD, AE &c.

COROLLARY 6. By means of the 47th proposition, any two sides of a right-angled triangle being given, the third side may be readily found; for if the two sides given are those which form the right angle, the sum of the squares on those two sides will be equal to the square on the third side; and if one of the given sides subtends the right angle, then the square on the third side is equal to the difference of the squares on the two given sides. This application of the 47th proposition to the purposes of finding the parts of right-angled triangles is of immense use, and is really the foundation of Trigonometry.

COROLLARY 7. The 47th proposition holds true, if, instead of squares, we construct any similar figures on the sides of the triangle, such as circles, equilateral triangles, &c., and may therefore be generalized as follows:—

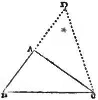
THEOREM. *If a triangle be right-angled,* any figure which is constructed upon the side subtending the right angle is equal in area to the sum of the similar figures constructed upon the sides which form the right angle.



SCHOLIUM. If three semicircles be described on the three sides of a right-angled triangle, the area of ABC will therefore be equal to the sum of the areas of ADB and BFC; if, now, from each we deduct the common segments AEB and BGC, we have the triangle ABC equal in area to the sum of two lunes ADBE and BFCG. This proposition was discovered by Hippocrates, and was the first instance of the determination geometrically of the area of a space entirely bounded by curved lines; it led him to believe that, by means of this proposition, he would be able to solve the problem, to determine geometrically the area of the circle.

PROPOSITION XLVIII.

THEOREM.—*If the square constructed upon one side* (BC) *of a triangle* (ABC) *be equal in area to the sum of the squares constructed upon the other two sides* (AC and AB), the angle BAC opposite to that side is a right angle.

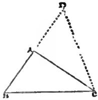


CONSTRUCTION. *From the point* A *draw* AD *at right angles to one of the sides* AC (*a*), *and equal to the other* AB (*b*); *and join* DC.

DEMONSTRATION. Because DA equals AB (*c*), therefore the square on DA equals the square on AB (*d*); and adding to each the square on AC, therefore the sum of the squares on DA and AC is equal to the sum of the squares on AB and AC (*e*). But the square on DC is equal in area to the sum of the squares on DA and AC (*f*), because DAC is a right angle (*c*); and the square on BC is equal in

(*a*) I. 11.
(*b*) I. 2.
(*c*) Const.
(*d*) I. 46, cor. 1.
(*e*) Ax. 2.
(*f*) I. 47.

area to the sum of the squares on AB and AC (*g*); therefore the square on DC is equal to the square on BC, and therefore DC is equal to BC (*h*). Then, because in the triangles DAC and BAC the sides AB and AD are equal, AC common to both, and the base DC equal to BC, therefore the angle DAC is equal to BAC (*i*); but DAC is a right angle (*c*), *therefore* BAC *is also a right angle.*



(*c*) Const.
(*g*) Hypoth.
(*h*) I. 46, cor. 2.
(*i*) I. 8.

SCHOLIUM. This proposition is the converse of the preceding one.

THE

ELEMENTS OF EUCLID.

BOOK II.

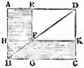
DEFINITIONS.

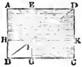
1. Every rectangle is said to be *contained* by any two of the straight lines which contain one of the right angles.

Scholium. As already explained in the scholium to the twenty-seventh definition in the former Book, a rectangle is designated as the *rectangle under the two lines* by which it is contained.

2. In any parallelogram, either of the parallelograms about the diagonal, together with the two complements, is called a *gnomon.*

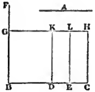
Scholium. Thus the shaded portions in the annexed figures, made up of the parallelogram HG and its complements AF and FC, or the parallelogram EK and the same complements, are both *gnomons.*





PROPOSITION I.

Theorem.—*If there be two straight lines* (A and BC), *one of which is divided into any number of parts* (BD, DE, EC), the rectangle under the two lines is equal in area to the sum of the rectangles under the undivided line (A) and the several parts of the divided line (BD, DE, EC).



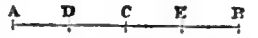
Construction. *From the point* B *draw* BF *perpendicular to* BC (*a*), *and make* BG *equal to* A (*b*); *through* G *draw* GH *parallel to* BC (*c*), *and through* D, E, C, *draw* DK, EL, CH, *parallel to* BG (*c*).

Demonstration. It is evident that the rectangle BH is equal to the sum of the rectangles BK, DL, EH; but the rectangle BH is the rectangle under A and BC, for BG is equal to A (*d*); and the rectangles BK, DL, EH, are respectively the rectangles under A and BD, A and DE, A and EC, for each of the lines BG, DK, EL, is equal to A (*e*). *Therefore the rectangle under*

(*a*) I. 11.
(*b*) I. 3.
(*c*) I. 31.
(*d*) Constr.
(*e*) I. 34 and Constr.

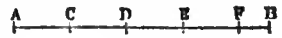
A and BC *is equal in area to the sum of the rectangles under* A *and* BD, A *and* DE, *and* A *and* EC.

Scholium. It is of considerable importance that the true relationship or connection between Geometry and Analysis be clearly understood before entering upon the second book of the Elements. The subject of Geometry is *magnitude;* that of Analysis, *i. e.*, Arithmetic and Algebra, is *number* and *quantity.* In order, therefore, to a just understanding of the connection between Geometry and Algebra, we must first obtain a clear notion of the connection between the subject of each, that is, between *magnitude* and *number.* Let us take a line AB of any given magnitude, and bisect it in C; and let us again bisect AC in D, and CB in E. We thus obtain four equal lines, which are together equal to the given line AB; and it is evident that, being all equal, any one of those lines



taken four times will be equal to the four lines, or to the original line AB; and we have thus two modes of expressing the magnitude of the line AB, namely, either by saying that it is equal to four lines each equal to AD, or by saying that it equals AB. That is, we may either view the line in its entirety, or we may—having first conceived it as being divided into any *number* of equal parts—view it as the magnitude of one of those parts repeated that *number* of times. Each of these parts is termed a *unit;* the process which we follow in order to determine the number of units in a given line is termed *measuring* that line; and if the unit is found to be contained any exact number of times in the given line, that is to say, if the unit added on to itself any certain definite number of times forms a line neither longer nor shorter than the given line, but exactly coinciding therewith, the unit is termed the *measure of the line;* and if the same unit is found to measure any other given line, so that being taken any other certain definite number of times, it forms another line exactly equal to the second given line, that unit is said to be the *common measure of* both the given lines, and those two lines are said to be *commensurable.* A unit may be arbitrarily determined, that is to say, we may assume a line of any length that we please for the purpose; but having once determined the magnitude that shall constitute the unit, that magnitude must be considered as fixed and unalterable. Thus, in any particular course of investigation, we may assume for the unit a line a yard in length, or a foot in length, or an inch in length; and having done so, we should express the *magnitude* of any line by the *number* of lines (each one yard, foot, or inch, as the case might be) which would form a line equal in magnitude to the given line. Here, then, we see the connection between number and magnitude; *number* may be regarded as the instrument through the medium of which we estimate and express *magnitude.* Were we not in possession of the common notion or idea of number, we could only express the magnitude of any given line by the actual exhibition of the line, or of another equal to it; but, having that idea, we are enabled to declare its magnitude by comparing it with another standard line (termed a unit), the magnitude of which is already familiar to us; and we thus make known the magnitude of the given line, by stating the result of that comparison, or the *number* of those units which the line is equal to.

We have hitherto assumed that the given line has, in every case, been equal to a certain number of units; but let us now suppose that, having arbitrarily fixed upon a unit, when we apply it to the given line, as AB, by cutting off successive portions AC, CD, DE, &c., each equal to the unit, we at length arrive at a remaining portion, as FB, which is less than the unit; how are we, in such a case, to determine the magnitude of the given line? The most obvious mode

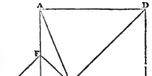


would be, to divide one of the units, as EF, by continually bisecting it, until the portion FB of the given line was found to be equal in length to some certain number of the minute equal parts into which we had so divided the unit; and, in every case in which this process could be carried out, the magnitude of the given line would be determined with the same exactness as if it had at once coincided with any given number of units; for we might regard any one of the small parts into which we had divided the original unit as a new unit, and express the length of the given line by stating the number of such lesser units contained in its length. Thus, if the original unit had been a foot, and on applying it to the given line we had found it contained four times, together with a remainder less than a foot; but that, on dividing the unit into twelve equal parts, the remainder was found to be precisely equal to eight of those twelfth-parts, we could, in such case, declare the length of the original line by stating it to be equal to that of 56 units, each a twelfth of a foot, or one inch, in length.

It might be regarded as almost self-evident, that this mode of measuring a line could always be adopted; that, in fact, whatever might be the comparative lengths of the unit AC and the remainder FB, by a sufficiently minute subdivision of the former, we could always arrive at some new unit which would be contained in the latter a certain definite number of times; that if, for instance, it was not found to be equal to any definite number of hundredth-parts, it might be of thousandths, or millionths, or even of some much more minute division. But it may be demonstrated that this frequently cannot be done; that, in fact, certain lines have no unit or common measure, however minute, by which they can be both divided without a remainder; and, when such is the case, they are said to be *incommensurable.* The following lemma is introduced as an instance.

LEMMA. *If two straight lines* (AB and BD) *are the side and diagonal of a square* (ABCD), they are incommensurable.

CONSTRUCTION. *From* BD *cut off* DE *equal to* AB (*a*); *through* E *draw* EF *perpendicular to* BD (*b*), *and produce it to cut* AB *in* F. *Join* AE.

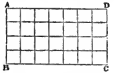


DEMONSTRATION. Because the triangle ADE is isosceles (*c*), the angles DAE and DEA are equal (*d*). Then, because DAF and DEF are both right angles (*c*), they are equal (*e*). Therefore if the equals DAE and DEA be taken from the equals DAF and DEF, the remaining angles FAE and FEA are equal (*f*). And because in the triangle AFE, the two angles FAE and FEA are equal, therefore the opposite sides AF and FE are equal (*g*). But in the triangle BEF the angles FBE and BFE are evidently each equal to half a right angle, therefore the opposite sides BE and FE are equal (*g*). Complete the square HE, and on its diagonal FB take FG equal to BE or AF (*a*); wherefore the excess BE of the diagonal BD beyond the side AB is contained twice in that side with a remainder GB; and GB being itself the excess of the diagonal BF beyond the side HB (of the square HE) is contained twice in that side with a remainder LB, which will again be contained twice in the side KB with a remainder; and this process of subdivision might be carried on with-

(*a*) I. 3.
(*b*) I. 11.
(*c*) Constr.
(*d*) I. 5.
(*e*) I. Ax. 11.
(*f*) I. Ax. 3.
(*g*) I. 6.

out limit, whence it is evident *that no common measure can be found for the side and diagonal of a square, but that they are incommensurable.*

Hitherto we have only mentioned one species of magnitude as being measured by number, namely lines; but every kind of magnitude, whether a surface, solid, or angle, may be so estimated, it being only necessary that the unit arbitrarily fixed upon should be of the same species or kind as the magnitude to be measured; that is to say, the unit for measuring surfaces must itself be a surface; for measuring solids, a solid; for angles, an angle. In these, as in the case of lines, the magnitude of the unit is entirely arbitrary; but it is convenient and usual to take, as the unit for measuring surfaces, a *square*, the length of whose side is equal to the linear unit employed for the measurement of lines; and for estimating the content of a solid, to employ a *cube*, any one of whose bounding-planes is equal to the square unit. Thus if a *linear foot* had been assumed as the measure of a line, the *square foot* would be employed as that of a surface, and the *cubic foot* as that of a solid.



Let ABCD be a rectangle whose two sides are commensurable, the side AB containing four units, and BC seven units; and let those sides be divided respectively into four and seven equal parts, and lines drawn through the points of division perpendicular to the divided side. Now it is evident that each of the portions into which the rectangle is thus divided is a square unit, and that as the area or magnitude of a rectangle is expressed by the number of units it contains, that of the rectangle ABCD will be found by estimating the number of squares into which it has been divided. We first perceive that the horizontal lines divide it into four equal portions, and next that each of these portions is again divided into seven equal parts by the vertical lines, so that the total number of subdivisions will be found by taking 7 four times, or multiplying seven by four; that is to say, that the number of square units in any rectangle will be found by multiplying together the two numbers which express in units the magnitude of its two contiguous sides. When the rectangle is a *square*, the number of units in each of the two contiguous sides being the same, we have to multiply that number by itself to obtain its magnitude in square units; and hence the term *square*, which is applied in Algebra and Arithmetic to the product of a number multiplied by itself. It is, however, important that the term square in Geometry should not be confounded with that in Arithmetic, for which end we speak of a square in Geometry as the square *on* a line, that in Arithmetic as the square *of* a number. It would, however, be better in the latter case to avoid altogether the use of the word square, and to substitute the expression *power of a number*. Thus if a represent the number of linear units contained in one side of a square, the product of a multiplied by itself, or a^2, will equal the number of square units in the square; and similarly, if b represent the number of linear units contained in one side of a rectangle, and c the number in the contiguous one, b multiplied by c, or $b \cdot c$, will express the number of square units in the rectangle.

It will thus be seen that, by means of this symbolism, we can represent and express the magnitude of any rectangle, and deduce algebraically all the properties investigated geometrically by Euclid. But when we attempt to substitute for b and c their numerical values, it may be found that the two magnitudes which they represent are incommensurable, in which case, as no common measure or unit can exist by means of which their lengths can be stated, no definite numerical value can be given to b and c, and therefore the magnitude of the rectangle cannot be *exactly* found arithmetically, although, in practice, by a minute subdivision of its sides, its magni-

tude may be determined within any amount of exactness that may be desired.

In considering the properties of lines algebraically, it is necessary that they should always be measured in the same direction, that is, either always from right to left, or from left to right; but if, in the same investigation, it is found necessary to measure one or more of the lines in a contrary direction to the others, the magnitude of such line or lines must be considered as having a negative value, and any algebraical quantity assumed as representing that value must have the minus sign (—) prefixed to it.

Having thus pointed out both the connection and the difference between number and magnitude, we shall append to such of the propositions in the second book as admit of being so demonstrated an algebraical investigation and proof.

Theorem. *If there be two numbers* (a and b), *one of which is divided into any number of parts* (m, n, p), the product of the two numbers is equal to the sum of the products of the undivided number (a) and the several parts of the divided number (m, n, p).

Demonstration. By the hypothesis—

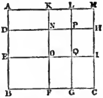
$$b = m + n + p.$$

Multiply these equal quantities by a, then

$$b \,.\, a = (m + n + p) \,.\, a, \text{ or}$$
$$a\,b = a\,m + a\,n + a\,p.$$

Corollary. *If two straight lines* (AB and BC) *be each of them divided into any number of parts* (AD, DE, EB, and BF, FG, GC), the rectangle under the two lines is equal in area to the sum of all the rectangles under all the parts of the one, taken separately with all the parts of the other.

For the rectangle under AB and BF is equal in area to the sum of the rectangles under BF and AD, BF and DE, and BF and EB; also the rectangle under AB and FG is equal in area to the sum of the rectangles under FG and AD, FG and DE, and FG and EB; and the rectangle under AB and GC is equal in area to the sum of the rectangles under GC and AD, GC and DE, and GC and EB. But the rectangle under AB and BC is equal in area to the sum of the rectangles under AB and BF, AB and FG, and AB and GC; *therefore it is equal in area to the sum of all the rectangles under all the parts of the one, taken separately with all the parts of the other.*



The above corollary, stated algebraically, will be as follows:—

If two numbers (a and b) *be each of them divided into any number of parts* (m, n, p, and q, r, s), the product of the two numbers is equal to the sum of the products of all the parts of the one, multiplied by all the parts of the other.

For by the hypothesis—

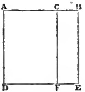
$$a = m + n + p, \text{ and}$$
$$b = q + r + s.$$

Then if we multiply the first member of the first equation by b, and the second member by its equal $(q + r + s)$, we have

$$a \,.\, b = (m + n + p) \,.\, (q + r + s), \text{ or}$$
$$a\,b = m\,q + n\,q + p\,q + m\,r + n\,r + p\,r + m\,s + n\,s + p\,s.$$

PROPOSITION II.

THEOREM.—*If a straight line* (AB) *be divided into any two parts* (in C), the rectangles under the whole (AB), and each of the parts (AC, CB), are together equal in area to the square on the whole line (AB).



CONSTRUCTION. *On* AB *construct the square* ADEB (*a*), *and through* C *draw* CF *parallel to* AD (*b*).

DEMONSTRATION. It is evident that the square AE is equal to the sum of the rectangles AF, CE; but because AD is equal to AB (*c*), the rectangle AF is the rectangle under AD, or AB and AC; and because CF is equal to AD (*d*), the rectangle CE is the rectangle under CF, or AB and CB. *Therefore the square* AE *on the whole line is equal in area to the rectangles* AF *and* CE *under the whole line and its parts.*

(*a*) I. 46.
(*b*) I. 31.
(*c*) Constr.
(*d*) I. 34.

SCHOLIUM. This proposition, algebraically expressed, is as follows:—

THEOREM. *If a number* (*a*) *be divided into any two parts* (*m*, *n*), the products of the whole number and each of the parts are together equal to the number multiplied by itself, or to the second power of that number.

DEMONSTRATION. By the hypothesis—

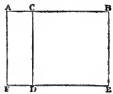
$$a = m + n.$$

Multiply these equal quantities by a, then

$$a \,.\, a = (m + n) \,.\, a, \text{ or}$$
$$a^2 = a\,m + a\,n.$$

PROPOSITION III.

THEOREM.—*If a straight line* (AB) *be divided into any two parts* (in C), the rectangle under the whole line (AB) and one of those parts (CB) is equal in area to the square on that part (CB) together with the rectangle under the two parts (AC and CB).



CONSTRUCTION. *On* CB *construct the square* CDEB (*a*), *and through* A *draw* AF *parallel to* CD (*b*) *until it meet* DE *produced in* F.

(*a*) I. 46.
(*b*) I. 31.
(*c*) Constr. and I. Def. 28.
(*d*) Constr.

DEMONSTRATION. It is evident that the rectangle AE is equal to the square CE, together with the rectangle AD; but the rectangle AE is the rectangle under AB and CB; for BE is equal to CB (*c*), and the square CE is on CB (*d*); and the rectangle AD is the rectangle under

AC and CB, for CD is equal to CB (*c*). *Therefore the rectangle under* AB *and* CB *is equal in area to the square on* CB, *together with the rectangle under* AC *and* CB.

Scholium. This proposition, algebraically expressed, is as follows:—

Theorem. *If a number* (a) *be divided into any two parts* (m, n), the product of the whole number and one of the parts is equal to the product of that part multiplied by itself, together with the product of the two parts.

Demonstration. By the hypothesis—

$$a = m + n.$$

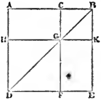
Multiplying these equal quantities by m, we have

$$a \,.\, m = (m + n) \,.\, m, \text{ or}$$

$$a\,m = m^2 + m\,n.$$

PROPOSITION IV.

Theorem.—*If a straight line* (AB) *be divided into any two parts* (in C), the square on the whole line is equal in area to the sum of the squares on the parts (AC and CB), together with twice the rectangle under the parts.



Construction. *On* AB *construct the square* ADEB (*a*), *and join* BD; *through* C *draw* CF *parallel to* AD (*b*), *and through* G *draw* HK *parallel to* AB (*b*).

Demonstration. Because the line BD intersects the parallel lines CF and AD, it forms the external angle BGC equal to the interior opposite angle BDA (*c*); and because the triangle ADB is isosceles (*d*), the angle ABD is equal to the angle BDA (*e*); therefore the angle BGC is equal to ABD, and the side CG to CB (*f*). And because CGKB is a parallelogram (*d*), the side GK is equal to CB, and BK to CG (*g*), it is therefore equilateral. Also because the line AB intersects the parallel lines CF and AD, it forms the alternate angles A and BCG equal (*c*), therefore the angle BCG is a right angle, and CGKB is a square on CB (*h*). And in the same manner it may be shown that the parallelogram HDFG is also a square on AC, because HG is equal to AC (*g*). Further AG and GE are together equal in area to double the rectangle under AC and CB, because AG is equal to GE (*i*), and AG is the rectangle under AC and CG, which is equal to CB. *Therefore the square* ADEB *on the whole line is equal in area to the sum of the squares* HF *and* CK *on the parts of the line, together with twice the rectangle under those parts.*

(*a*) I. 46.
(*b*) I. 31.
(*c*) I. 29.
(*d*) Constr.
(*e*) I. 5.
(*f*) I. 6.
(*g*) I. 34.
(*h*) I. Def. 28.
(*i*) I. 43.

Scholium. This proposition, algebraically expressed, is as follows:—

THEOREM. *If a number (a) be divided into any two parts (m, n), the second power of the whole number is equal to the sum of the second powers of the parts, together with twice the product of the parts.*

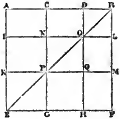
DEMONSTRATION. By the hypothesis—

$$a = m + n$$

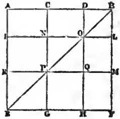
Squaring both sides of the equation—

$$a^2 = (m + n)^2, \text{ or}$$
$$a^2 = m^2 + 2\,m\,n + n^2.$$

COROLLARY 1. *If a straight line* (AB) *be divided into any number of segments* (AC, CD, DB), the square on the whole line is equal in area to the sum of the squares upon the segments, together with twice the rectangle under each pair of segments.



DEMONSTRATION. *On* AB *construct the square* AEFB, *and join* EB; *through* C *and* D *draw* CG *and* DH *parallel to* AE, *and through* P *and* O *draw* KM *and* IL *parallel to* AB. It may then be shown, by similar reasoning to that in the foregoing proposition, that the squares KG, NQ, and DL are respectively constructed on AC, CD, and DB; and further, that the rectangles IP and PH are equal to twice the rectangle under AC and CD, that the rectangles CO and OM are equal to twice the rectangle under CD and DB, and that the rectangles AN and QF are equal to twice the rectangle under AC and DB. *It is therefore evident that the square on* AB *is equal to the sum of the squares upon the segments* AC, CD, DB, *together with twice the rectangles under each pair* AC *and* CD, CD *and* DB AC *and* DB.

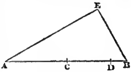


COROLLARY 2. *If a straight line* AB *be divided into three parts*, the squares on the double segments AD, CB, together with twice the rectangle under the extreme segments AC and DB, are equal in area to the squares on the whole line AB and the middle segment CD.

For it is evident that the squares IH and CM, together with the rectangles AN and QF, are equal in area to the square AEFB, together with the square NQ.

COROLLARY 3. *If from either end of the hypotenuse* (AB) *of a right-angled triangle parts be cut off equal to the adjacent sides*, the square on the middle segment (CD) thus formed is equal in area to twice the rectangle under the extreme segments (AC and DB).

DEMONSTRATION. For the straight line AB being divided into three parts, the squares on AD and CB, together with twice the rectangle under AC and DB, are equal in area to the squares on AB and CD (*a*). But the squares on AE and EB, or their equals AD and CB, are equal in area to the square on AB (*b*). Therefore, taking these equals from the former equals, *we have twice the rectangle under* AC *and* DB *equal in area to the square on* CD (*c*).



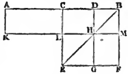
(*a*) Cor. 2.
(*b*) I. 47.
(*c*) I. Ax. 3.

COROLLARY 4. From the demonstration, it is manifest that the parallelograms about the diagonal of a square are also squares.

PROPOSITION V.

THEOREM.—*If a straight line* (AB) *be bisected* (in C), *and also cut into two unequal parts* (in D), the rectangle under the unequal parts (AD and DB), together with the square on the line (CD), between the points of section, is equal in area to the square on half the line (CB).

CONSTRUCTION. *On* CB *construct the square* CEFB (*a*), *and join* EB; *through* D *draw* DG *parallel to* CE (*b*), *through* H *draw* KM *parallel to* AB (*b*), *and through* A *draw* AK *parallel to* CL (*b*).



(*a*) I. 46.
(*b*) I. 31.
(*c*) I. 43.
(*d*) Hypoth.
(*e*) Ax. 2.
(*f*) II. 4, cor. 4.
(*g*) I. 34.

DEMONSTRATION. In the parallelogram CF the complements CH and HF are equal (*c*); and because AC is equal to CB (*d*), AL is equal to CM; therefore the rectangle AH is equal in area to the gnomon CMG (*e*). To each of these add the square LG, and the rectangle AH together with the square LG is equal in area to the whole square CEFB (*e*). But, because DH is equal to DB (*f*), AH is the rectangle under AD and DB; and because LH is equal to CD (*g*), LG is the square on CD, and CEFB is the square on CB. Therefore *the rectangle under* AD *and* DB, *together with the square on* CD, *is equal in area to the square on* CB.

SCHOLIA. 1. This proposition, algebraically expressed, is as follows:—

THEOREM. *If a number* (*a*) *be divided into two equal parts* (*m*, *n*), *and also into two unequal parts* (*p*, *q*), the product of the unequal parts, together with the second power of the difference between either of the unequal parts and equal parts, is eqnal to the second power of either of the equal parts.

DEMONSTRATION. Let p be the greater of the unequal parts, and d the difference between p and m; then by the hypothesis—

$$p = m + d, \text{ and}$$
$$q = m - d.$$

Then if we multiply the first member of the first equation by q, and the second member by its equal $(m - d)$, we have

$$p \cdot q = (m + d) \cdot (m - d) = m^2 - d^2,$$

and transposing, we obtain

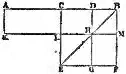
$$p\,q + d^2 = m^2.$$

2. The fifth proposition may be otherwise expressed, as follows:—

THEOREM. The square on half the sum of two lines is equal in area to the rectangle under them, together with the square on half their difference.

For if AD and DB be considered the two lines, AC is half their sum, CD half their difference, and AH the rectangle under them.

COROLLARY 1. The rectangle under the sum and difference of two lines (CB and CD) is equal in area to the difference of the squares on those lines.



DEMONSTRATION. For the gnomon LBG is the difference between the square on CB and the square on CD. But the gnomon LBG is equal in area to the rectangle under AD and DB (*a*); and because AC is equal to CB, therefore AD is equal to the sum of CB and CD, and DB is equal to their difference.

(*a*) II. 5.

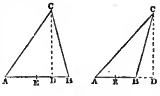
SCHOLIA. 1. The foregoing corollary, expressed algebraically, forms a theorem of the utmost importance.

The product of the sum and difference of two numbers (a and b) is equal to the difference of their second powers.

$$\text{For } (a+b) \cdot (a-b) = a^2 + ab - ab - b^2 = a^2 - b^2.$$

2. When a line is divided into two parts, the rectangle under those parts is a *maximum*, or the greatest possible, when the line is bisected; and the sum of the squares on those two parts is then a *minimum*, or the least possible.

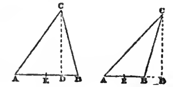
COROLLARY 2. *If a perpendicular* (CD) *be drawn from the vertex of a scalene triangle* (ABC), the difference of the squares on the sides (AC and CB) is equal in area to twice the rectangle under the base (AB) and the distance (DE) of its middle point from the perpendicular.



DEMONSTRATION. For the difference of the squares on the sides AC and CB is equal in area to the difference of the squares on the segments of the base AD and DB (*a*), and therefore to the rectangle under the sum and difference of the segments AD and DB (*b*); but when the perpendicular falls within the triangle, the base AB is equal to their sum, and the distance DE of its middle point from the perpendicular to half their difference; therefore *the difference of the squares on the sides* AC *and* CB *is equal in area to twice the rectangle under* AB *and* DE. And when the perpendicular falls without the triangle, the base AB is equal to the difference of the segments AD and BD, and the distance DE to half their sum; and therefore, in this case, *the difference of the squares on the sides* AC *and* CB *is equal in area to twice the rectangle under* AB *and* DE.

(*a*) I. 47, cor. 1.
(*b*) II. 5, cor. 1.

SCHOLIUM. From the foregoing corollary we derive the following rule for finding the area of a triangle of which the three sides are given. Divide the difference between the second power of the two sides of the triangle by twice the base; add half the base to the quotient, and subtract the second power of the sum from the second power of the greater side; the remainder is the second power of the perpendicular CD. Then half the product of the perpendicular CD multiplied by the base AB is the area of the triangle.



The foregoing rule, expressed algebraically, is as follows:—

$$AC^2 - \left(\frac{AC^2 - CB^2}{2 . AB} + \frac{AB}{2}\right)^2 = CD^2,$$

and may be demonstrated algebraically in the following way:—

$$CD^2 = AC^2 - AD^2 \text{ (I. 47)}.$$

Then, by the preceding corollary,

$$AC^2 - CB^2 = 2 . AB . DE,$$

and dividing both sides by 2 . AB,

$$\frac{AC^2 - CB^2}{2 . AB} = DE.$$

Then, since $AD = \frac{AB}{2} + DE$, substituting the above value of DE, we have

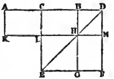
$$AD = \frac{AC^2 - CB^2}{2 . AB} + \frac{AB}{2},$$

which value of AD being substituted in the equation above, gives

$$CD^2 = AC^2 - \left(\frac{AC^2 - CB^2}{2 . AB} + \frac{AB}{2}\right)^2.$$

PROPOSITION VI.

THEOREM.—*If a straight line* (AB) *be bisected* (in C), *and produced to any point* (D), the rectangle under the whole line thus produced (AD) and the produced part (BD), together with the square on half the line bisected (CB), is equal in area to the square on the straight line (CD) which is made up of the half and the produced part.



CONSTRUCTION. *On* CD *construct the square* CEFD (*a*), *and join* ED (*b*); *through* B *draw* BG *parallel to* CE (*b*), *through* H *draw* KM *parallel to* AD (*b*), *and through* A *draw* AK *parallel to* CL (*b*).

DEMONSTRATION. In the parallelogram CF the complements CH and HF are equal (*c*), and because AC is equal to CB (*d*), AL is equal to CH; therefore AL is equal to HF (*e*). To each of these add the parallelogram CM, and the rectangle AM is equal in area to the gnomon LDG (*f*); and to each of these add the square LG, and the rectangle AM

(*a*) I. 46.
(*b*) I. 34.
(*c*) I. 43.
(*d*) Hypoth.
(*e*) Ax. 1.
(*f*) Ax. 2.

together with the square LG is equal in area to the whole square CEFD (f). But because DM is equal to BD (g), AM is the rectangle under AD and DB; and because LH is equal to CB (h), LG is the square on CB, and CEFD is the square CD. Therefore *the rectangle under* AD *and* DB, *together with the square on* CB, *is equal in area to the square on* CD.

(f) Ax. 2.
(g) II. 4, cor. 4.
(h) I. 34.

SCHOLIA. 1. This proposition, algebraically expressed, is as follows:—

THEOREM. *If a number* (a) *be divided into two equal parts* (m, n), *and any other number* (p) *be taken*, the product of the sum of the two numbers (a and p) and of the number so taken (p), together with the second power of half the number divided (m or n), is equal to the second power of the sum of half that number and the number taken (m and p).

DEMONSTRATION. By the hypothesis—

$$2\,m = a,$$

therefore

$$(a+p)\,.\,p+m^2=(2\,m+p)\,.\,p+m^2=2\,m\,p+p^2+m^2=m^2+2\,m\,p+p^2.$$

But

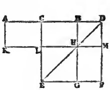
$$m^2+2\,m\,p+p^2=(m+p)^2,$$

therefore

$$(a+p)\,.\,p+m^2=(m+p)^2.$$

2. The fifth and sixth propositions are really identical; for it has been already shown that the fifth may be otherwise expressed, as in the second scholium annexed to that proposition, and the sixth may likewise be so expressed; for if AD and BD be considered the two lines, CD is half their sum, CB half their difference, and AM the rectangle under them.

COROLLARY 1. *If three lines are in arithmetical proportion*, the rectangle under the extremes, together with the square on the common difference, is equal in area to the square on the mean.



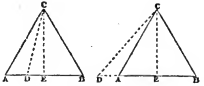
DEMONSTRATION. For the lines AD, CD, and BD are in arithmetical progression, having a common difference because AC equals CB. And *the rectangle under the extremes* AD *and* BD, *together with the square on the common difference* CB, *is equal in area to the square on the mean* CD (a).

(a) II. 6.

SCHOLIUM. Three lines are said to be in *arithmetical proportion* when the difference between the first and second is equal to the difference between the second and third, which difference is then termed the *common difference.*

COROLLARY 2. *If a straight line* (CD) *be drawn from the vertex of an isosceles triangle* (ABC) *to any point in the base or the base produced*, the rectangle under the segments of the base (AD and DB) is equal in area to the difference between the square on this line (CD) and the square on either side of the triangle (AC or BC).

DEMONSTRATION. *Bisect the base* AB *in* E (*a*), *and draw the straight line* CE. The rectangle under AD and DB is equal in area to the difference between the squares on AE and DE (*b*); add to both these squares the square on the perpendicular CE, and *the rectangle under* AD *and* DB *is equal in area to the difference between* the sum of the squares on AE and CE, or *the square on* AC (*c*), *and* the sum of the squares on DE and CE, or *the square on* CD (*c*).

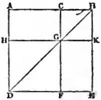


(*a*) I. 10.
(*b*) II. 5 or 6.
(*c*) I. 47.

PROPOSITION VII.

THEOREM.—*If a straight line* (AB) *be divided into any two parts*, the sum of the square on the whole line (AB) and the square on either segment (CB) is equal in area to double the rectangle under the whole line and that segment, together with the square of the other segment (AC)

CONSTRUCTION. *On* AB *construct the square* ADEB (*a*), *and join* DB; *through* C *draw* CF *parallel to* AD (*b*), *and through* G *draw* HK *parallel to* AB (*b*).



DEMONSTRATION. The square on AB is equal in area to the rectangles AK and GE together with the square HF; add to both the square CK, and the sum of the squares AE and CK is equal in area to the rectangles AK and CE together with the square HF (*c*). But because CB is equal to BK (*d*), AK is the rectangle under AB and CB; and because BE is equal to AB (*e*), CE is also equal to the rectangle under AB and CB; and because HG is equal to AC (*f*), HF is the square on AC. Therefore *the sum of the square on* AB *and the square on* CB *is equal in area to double the rectangle under* AB *and* CB *together with the square on* AC.

(*a*) I. 46.
(*b*) I. 31.
(*c*) Ax. 1.
(*d*) II. 4, cor. 4.
(*e*) I. def. 28.
(*f*) I. 34.

SCHOLIUM. This proposition, algebraically expressed, is as follows:—

THEOREM. *If a number* (*a*) *be divided into two parts* (*m*, *n*), the sum of the second power of the whole number (*a*) and the second power of either of the parts (*m*) is equal to twice the product of the whole number and that part, together with the second power of the other part (*n*).

DEMONSTRATION. By the hypothesis—

$$m = a - n,$$

therefore

$$a^2 + m^2 = a^2 + (a - n)^2$$
$$= a^2 + a^2 - 2an + n^2$$
$$= 2a^2 - 2an + n^2$$
$$= 2a(a - n) + n^2,$$

and substituting for $(a - n)$ its equal m, we have

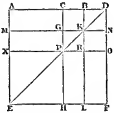
$$a^2 + m^2 = 2am + n^2.$$

COROLLARY. If we consider AB and CB as two independent lines, and AC as their difference, the foregoing proposition may be thus expressed:—

THEOREM. The sum of the squares on any two lines is equal in area to twice the rectangle under them, together with the square on their difference.

PROPOSITION VIII.

THEOREM.—*If a straight line* (AB) *be divided into any two parts*, the square on the sum of the whole line (AB) and either segment (BC) is equal in area to four times the rectangle under the whole line and that segment (AB and BC), together with the square on the other segment (AC).



CONSTRUCTION. *Produce* AB *until* BD *is equal to* CB (*a*). *On* AD *construct the square* AEFD (*b*), *and join* ED; *through* C *and* B *draw* CH *and* BL *parallel to* AE (*c*), *and through* K *and* P *draw* MN *and* XO *parallel to* AD (*c*).

DEMONSTRATION. Because GK is equal to CB (*d*), CB to BD (*e*), and BD to KN (*d*), GK is equal to KN (*f*), and therefore the rectangle GL is equal to the rectangle KF; and because AK and KF are complements, they are equal (*g*), therefore AK is equal to GL (*f*). Because GK is equal to KN, therefore GR is equal to BN (*h*); and because MP and PL are complements, they are equal (*g*); adding these equals together, the rectangle MP together with the square BN is equal in area to the rectangle GL (*i*), and therefore to its equal the rectangle AK (*f*). Therefore AK, GL, and KF, together with MP and BN, are equal in area to four times AK; but AK, GL, and KF, together with MP and BN, and the square XH, make up the whole square AEFD; therefore the square AEFD is equal in area to four times AK together with XH (*f*). But because BK is equal to BD (*k*), and BD to CB (*e*), BK is equal to CB (*f*), and there-

(*a*) I. 3 and Post. 2.
(*b*) I. 46.
(*c*) I. 31.
(*d*) I. 34.
(*e*) Constr.
(*f*) Ax. 1.
(*g*) I. 43.
(*h*) I. 46, cor. 1.
(*i*) Ax. 2.
(*k*) II. 4, cor. 4.

fore AK is the rectangle under AB and CB; and because XP is equal to AC (*d*), XII is the square on AC. Therefore *the square on the sum of* AB *and* BD *is equal in area to four times the rectangle under* AB *and* CB, *together with the square on* AC.

Scholium. The foregoing proposition, algebraically expressed, is as follows:—

Theorem. *If a number* (a) *be divided into two parts* (m, n), the second power of the sum of the whole number and either of the parts ($a + m$) is equal to four times the product of the whole number (a) and that part (m), together with the second power of the other part (n).

Demonstration. By the hypothesis—

$$m = a - n,$$

therefore

$$(a + m)^2 = (2a - n)^2 = 4a^2 - 4an + n^2 = 4a(a - n) + n^2,$$

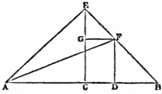
and substituting for $(a - n)$ its equal m, we have

$$(a + m)^2 = 4am + n^2.$$

PROPOSITION IX.

Theorem.—*If a straight line* (AB) *be bisected* (in C), *and also cut into two unequal parts* (in D), the sum of the squares on the unequal parts (AD and DB) is equal in area to double the sum of the square on half the line (AC) and the square on the line between the points of section (CD).

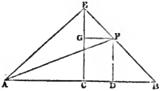
Construction. *From the point* C *draw* CE *perpendicular* (*a*) *and equal to* AC (*b*), *and join* EA *and* EB; *through* D *draw* DF *parallel* to CE (*c*), *and through* F *draw* GF *parallel to* AB (*c*); *and join* AF.



(*a*) I. 11.
(*b*) I. 2.
(*c*) I. 31.
(*e*) Constr.
(*f*) I. 32, cor. 4.
(*g*) I. 29.

Demonstration. Because in the triangle CAE the sides AC and CE are equal and the angle ACE a right angle (*e*), the angles CAE and CEA are each equal to half a right angle (*f*); and in the same manner it may be shown that the angles CEB and CBE are likewise each equal to half a right angle; therefore AEB is a right angle. Then because GF is parallel to CD (*e*), the angle EGF is equal to the interior and opposite angle ECB (*g*), therefore EGF is a right angle; and in the triangle EGF

the angle EGF being a right angle, and the angle GEF half a right angle, the other angle GFE is also equal to half a right angle (*h*), and these angles being equal, the sides opposite to them, GE and GF, are also equal (*i*); and in the same manner it may be shown that FD and DB are also equal to each other. Again, because in the triangle CAE, AC and CE are equal, and the angle ACE is right, the square on AE is double the square on AC (*k*); and because in the triangle GEF, GE and GF are equal, and the angle EGF is right, the square on EF is double the square on GF (*k*); but CD is equal to GF (*l*), therefore the square on EF is double the square on CD; and therefore the squares on AE and EF, taken together, are double the squares on AC and CD taken together. But because in the triangle EAF the angle AEF is a right angle, the square on AF is equal in area to the squares on AE and EF taken together (*k*); therefore the square on AF is double the squares on AC and CD taken together. But because in the triangle DAF the angle ADF is a right angle, the square on AF is equal in area to the squares on AD and DF taken together (*k*); therefore the sum of the squares on AD and DF is double the sum of the squares on AC and CD; but DF and DB are equal, therefore *the sum of the squares on* AD *and* DB *is equal in area to double the sum of the square on the line* AC, *and the square on the line* CD.



(*h*) I. 32, cor. 2.
(*i*) I. 6.
(*k*) I. 47.
(*l*) I. 34.

SCHOLIA. 1. The foregoing proposition, algebraically expressed, is as follows:—

THEOREM. *If a number* (*a*) *be divided into two equal parts* (*m*, *n*), *and also into two unequal parts* (*p*, *q*), the sum of the second powers of the unequal parts (*p* and *q*) is equal to twice the sum of the second powers of half the line (*m*), and of the difference between either of the unequal parts and equal parts.

DEMONSTRATION. Let p be the greater of the unequal parts, and d the difference between p and m; then by the hypothesis—

$$p = m + d, \text{ and } q = n - d,$$

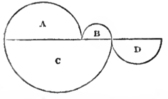
therefore

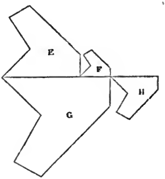
$$p^2 + q^2 = (m + d)^2 + (n - d)^2 = m^2 + 2\,m\,d + d^2 + n^2 - 2\,n\,d + d^2.$$

But by the hypothesis $m = n$, therefore

$$p^2 + q^2 = 2\,m^2 + 2\,d^2.$$

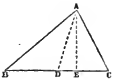
2. The foregoing proposition holds true if, instead of squares, we construct any *similar* figures on the respective lines, such as similar segments





of circles, or any similar irregular figures; thus the sum of the segments C and D is double the sum of the segments A and B, and the sum of the figures G and H is double the sum of the figures E and F.

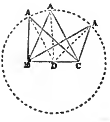
COROLLARY 1. THEOREM. *In a triangle* (ABC) *if a line* (AD) *be drawn from its vertex to bisect its base* (BC), the sum of the squares on the two sides (AB and AC) is equal in area to double the sum of the squares on half the base (BD) and on the bisecting line (AD).



DEMONSTRATION. *Draw* AE *perpendicular to* BC (*a*). Then because AEB is a right angle, the square on AB is equal in area to the sum of the squares on BE and AE (*b*); and because AEC is a right angle, the square on AC is equal in area to the sum of the squares on EC and AE (*b*); therefore the squares on AB and AC, taken together, are equal in area to the squares on BE and EC and twice the square on AE taken together (*c*). But because the line BC is cut equally in D and unequally in E, the squares on BE and EC, taken together, are equal to twice the squares on BD and DE taken together (*d*); therefore the squares on AB and AC, taken together, are equal in area to double the sum of the squares on BD, DE, and AE. But the square on AD is equal in area to the squares on DE and AE taken together (*b*); therefore *the sum of the squares on* AB *and* AC *is equal in area to double the sum of the squares on* BD *and* AD.

(*a*) I. 12.
(*b*) I. 47.
(*c*) Ax. 2.
(*d*) II. 9.

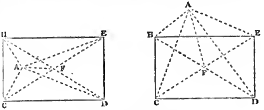
COROLLARY 2. THEOREM. *If from the middle point* (D) *of a finite straight line* (BC) *as a center, a circle be described, and lines be drawn from any point* (A) *in its circumference to the extremities of the line* (BC), the sum of the squares on those lines is always the same, and equal in area to double the sum of the squares on the radius (AD) and half the given line (BC).



For the base of the triangle ABC is bisected by the radius AD (*a*), and therefore the sum of the squares on its two sides AB and AC is equal in area to double the sum of the squares on half the base BD, and on the bisecting line AD (*b*).

(*a*) Constr.
(*b*) II. 9, cor. 1.

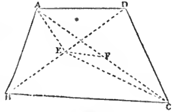
Corollary 3. Theorem. *If lines be drawn from any point* (A) *to the opposite angles of a rectangle* (BCDE), the sum of the squares on the lines (AC and AE) to one pair of opposite angles is equal in area to the sum of the squares on the lines (AB and AD) to the other pair of opposite angles.



Demonstration. *Draw the diagonals* BD *and* CE, *and join their point of intersection* F *and* A. Because in the triangles BEC and CDB the side BE is equal to CD (*a*), the side BC common to both, and the angle EBC equal to the angle BCD, the opposite sides CE and BD are equal (*b*); and because the diagonals bisect each other in F (*c*), BF and CF are equal. Then in the triangle ABD, the sum of the squares on the two sides AB and AD is equal in area to double the sum of the squares on half the base BF, and on the bisecting line AF (*d*); and in the triangle ACE the sum of the squares on the two sides AC and AE is equal in area to double the sum of the squares on half the base CF, and on the bisecting line AF (*d*); therefore *the sum of the squares on* AB *and* AD *is equal in area to the sum of the squares on* AC *and* AE (*e*).

(*a*) I. 34.
(*b*) I. 4.
(*c*) I. 34, cor. 3.
(*d*) II. 9, cor. 1.
(*e*) Ax. 1.

Corollary 4. Theorem. The squares on the sides of a quadrilateral figure (ABCD) are together equal in area to the sum of the squares on the diagonals, together with four times the square on the line joining their middle points (E and F).



Demonstration. *Draw* AE *and* EC. In the triangle ABD the sum of the squares on AB and AD is equal in area to twice the sum of the squares on AE and ED (*a*); and in the triangle CDB the sum of the squares on CD and CB is equal in area to twice the sum of the squares on EC and ED (*a*). But the sum of the squares on AE and EC is equal to twice the squares on EF and FC (*a*); therefore the sum of the squares on the sides BA, AD, DC, and CB, taken together, is equal in area to four times the square on ED, four times the square on FC, and four times the square on EF taken together, or to the sum of the squares on BD and AC together with four times the square on EF.

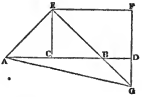
(*a*) II. 9, cor. 1.

Corollary 5. Theorem. The squares on the sides of a parallelogram are together equal in area to the sums of the squares on its diagonals.

For in this case the points E and F coincide, and therefore the line EF vanishes, whence the truth of this proposition is evident from the preceding corollary.

PROPOSITION X.

Theorem.—*If a straight line* (AB) *be bisected* (in C), *and produced to any point* (D), the square on the whole line thus produced (AD), together with the square on the produced part (BD), is equal in area to double the square on half the line bisected (CB), together with double the square on the straight line (CD), which is made up of the half and the produced part.



Construction. *From the point* C *draw* CE *perpendicular* (*a*) *and equal to* AC (*b*), *and join* EA *and* EB; *through* E *draw* EF *parallel to* AB (*c*), *and through* D *draw* DF *parallel to* CE (*c*). *Then because the line* EF *meets the parallels* CE *and* DF, *the angles* CEF *and* F *are together equal to two right angles* (*d*); *therefore the angles* BEF *and* F *are together less than two right angles, and the lines* EB *and* FD, *if produced, must meet* (*e*); *let them meet in* G, *and join* GA.

(*a*) I. 11.
(*b*) I. 2.
(*c*) I. 31.
(*d*) I. 29.
(*e*) Theor. attached to I. 29.
(*f*) Constr.
(*g*) I. 32, cor. 4.
(*h*) I. 29.
(*i*) I. 15.
(*k*) I. 32, cor. 2.
(*l*) I. 6.
(*m*) I. 34.
(*n*) I. 47.

Demonstration. Because in the triangle CAE the sides AC and CE are equal, and the angle ACE a right angle (*f*), the angles CAE and CEA are each equal to half a right angle (*g*); and in the same manner it may be shown that the angles CEB and CBE are likewise each equal to half a right angle; therefore AEB is a right angle. Then because GF is parallel to CE (*f*), the angle GDB is equal to the alternate angle ECB (*h*), therefore GDB is a right angle; also the angle DBG is equal to the vertical angle CBE (*i*), therefore DBG is half a right angle; therefore in the triangle DBG the angle GDB being a right angle, and the angle DBG half a right angle, the other angle DGB is also equal to half a right angle (*k*), and these angles being equal, the sides opposite to them DG and BD are also equal (*l*); and since the angle F is a right angle, being equal to the opposite angle ECB (*m*), it may in the same manner be shown that EF and FG are also equal to each other. Again, because in the triangle CAE, AC and CE are equal, and the angle ACE is right, the square on AE is double the square on AC (*n*); and because in the triangle GEF, GF and FE are equal, and the angle F is right, the square on GE is double the square on EF (*n*); but CD is equal to EF (*m*),

therefore the square on GE is double the square on CD; and therefore the squares on AE and GE, taken together, are double the squares on AC and CD taken together. But because in the triangle EAG the angle AEG is a right angle, the square on AG is equal in area to the squares on AE and EG taken together (n); therefore the square on AG is double the squares on AC and CD taken together. But because in the triangle DAG the angle ADG is a right angle, the square on AG is equal in area to the squares on AD and DG taken together (n); therefore the sum of the squares on AD and DG is double the sum of the squares on AC and CD; but DG and BD are equal, therefore *the sum of the squares on* AD *and* BD *is equal in area to double the sum of the square on the line* AC, *and the square on the line* CD.

SCHOLIA. 1. The foregoing proposition, algebraically expressed, is as follows:—

THEOREM. *If a number* (a) *be divided into two equal parts* (m, n), *and any other number* (p) *be taken*, the second power of the sum of the two numbers (a and p), together with the second power of the number so taken (p), is equal to twice the second power of half the number so divided (m or n), together with twice the second power of the sum of half that number and the number taken (m and p).

DEMONSTRATION. By the hypothesis—

$$2\,m = a,$$

therefore

$$\begin{aligned}(a+p)^2 + p^2 &= (2\,m+p)^2 + p^2\\ &= 4\,m^2 + 4\,m\,p + 2\,p^2\\ &= 2\,m^2 + 2\,(m^2 + 2\,m\,p + p^2).\end{aligned}$$

But

$$2\,(m^2 + 2\,m\,p + p^2) = 2\,(m+p)^2,$$

therefore

$$(a+p)^2 + p^2 = 2\,m^2 + 2\,(m+p)^2.$$

2. It will be seen that the hypothesis of the ninth proposition is the same as that of the fifth, and the hypothesis of the tenth as that of the sixth. There is, in fact, the same relationship between the ninth and tenth as has already been shown to exist between the fifth and sixth, the two former propositions being identical.

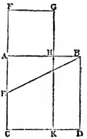
COROLLARY. THEOREM. The sum of the squares on two lines is equal in area to double the square on half their sum, together with double the square on half their difference.

For if AD and DB be considered the two lines, CD is half their sum and CP. half their difference.

PROPOSITION XI.

Problem.—To divide a given *finite straight line* (AB) into two parts, so that the rectangle under the whole line and one segment shall be equal in area to the square on the other segment.

Solution. *On* AB *construct the square* ABDC (*a*); *bisect* AC *in* E (*b*), *and join* BE; *produce* CA *to* F, *make* EF *equal to* EB (*c*), *and in the given line* AB *take* AH *equal to* AF (*d*); *and* AB *is divided in* H *so that the rectangle under* AB *and* BH *is equal in area to the square on* AH. *Through* H *draw* GK *parallel to* FC (*e*), *and through* F *draw* FG *parallel to* AB (*e*).



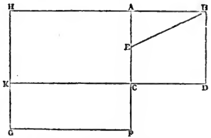
(*a*) I. 46.
(*b*) I. 10.
(*c*) I. 2.
(*d*) I. 3.
(*e*) I. 31.
(*f*) Solution.
(*g*) II. 6.
(*h*) I. 47.
(*i*) Ax. 1.
(*k*) Solution and I. 34.
(*l*) Ax. 3.
(*m*) I. Def. 28.

Demonstration. Because CA is bisected in E and produced to F (*f*), the rectangle under CF and AF, together with the square on AE, is equal in area to the square on EF (*g*), and therefore to the square on EB, which is equal to EF (*f*). But the square on EB is equal in area to the square on AB together with the square on AE (*h*), therefore the rectangle under CF and AF, together with the square on AE, is equal in area to the square on AB together with the square on AE (*i*); and taking away the common square AE, the rectangle under CF and AF is equal in area to the square on AB; but AD is the square on AB, and because AF and FG are equal (*k*), the rectangle CG is the rectangle under CF and AF. Then if the common rectangle CH be taken from the equals AD and CG, the remainders HD and AG are equal (*l*); but HD is the rectangle under AB and BH, for BD is equal to AB (*m*), and AG is the square on AH, because AH and AF are equal and the angle HAF is a right angle (*m*). Therefore *the line* AB *is divided in* H, *so that the rectangle under* AB *and* BH *is equal in area to the square on* AH.

Scholium. There is a second point in AB produced that will fulfil the requirements of the foregoing proposition, which may therefore be expressed as follows:—

Problem. In a given *finite straight line* (AB), *or its continuation*, to find a point, such that the rectangle under the whole line and the segment between one of its extremities and that point, shall be equal in area to the square on the segment between its other extremity and that point.

The solution of the second case of this problem is precisely the same as



that of the former, excepting only that AH is to be taken equal to AF on the production of AB instead of on the line itself; and the demonstration is the same with the exception that the common rectangle CH is to be added to the equals AD and CG instead of being deducted from them.

SCHOLIA. 1. The foregoing proposition involves the theory of the quadratic equation, and affords an admirable example of the application of algebra to geometry. The problem, algebraically expressed, is as follows:—

To divide any quantity (b) so that the product of the quantity with one of its parts shall be equal to the second power of the other part.

Let x be the value of one of the parts; then $(b - x)$ is the value of the other one, and the question requires that

$$x^2 = b \,.\, (b - x),$$

performing the multiplication

$$x^2 = b^2 - b\,x,$$

transposing

$$x^2 + b\,x = b^2,$$

adding $\frac{b^2}{4}$ to both sides,

$$x^2 + b\,x + \frac{b^2}{4} = b^2 + \frac{b^2}{4} = \frac{5\,b^2}{4},$$

extracting the square root

$$x + \frac{b}{2} = \frac{b}{2}\sqrt{5},$$

and again transposing

$$x = \frac{b}{2}(\sqrt{5} - 1) = b\,\frac{\sqrt{5} - 1}{2}.$$

Then b being taken as the value of the line AB, x will represent the value of AH in the first case of the foregoing problem.

In the second case, since x (or AH) is measured in an opposite direction to b (or AB), its sign is changed (as explained in the scholium to the first proposition); and consequently we have x equal one part, and $(b + x)$ equal the other part, and repeating the steps above, we obtain

$$x = b\,\frac{\sqrt{5} + 1}{2},$$

which is the value of AH in the second case, when H is taken in the production of AB.

We shall now proceed to show the bearing of this proposition upon the quadratic equation.

A *quadratic equation* is an expression containing the second power of the unknown quantity, and which can be reduced to the form

$$c\,x^2 + d\,x + e = 0. \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A.)}$$

We shall in the first place briefly investigate the value of x in this equation, and then point out the connection of the process with the foregoing problem.

Dividing the above equation by c, the coefficient of the first term, we obtain

$$x^2 + \frac{d}{c}x + \frac{e}{c} = 0,$$

and putting p for $\frac{d}{c}$, and q for $\frac{e}{c}$, we have

$$x^2 + p\,x + q = 0. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B.)}$$

Let a be a quantity, such that being substituted for x in this expression, it shall fulfil the conditions of the equation by making the first member equal 0; then we shall have

$$a^2 + p\,a + q = 0,$$

transposing

$$q = -\,a^2 - p\,a,$$

substituting this value of q in equation (B), we obtain

$$x^2 + p\,x - a^2 - p\,a = 0,$$

and again transposing

$$x^2 - a^2 + p\,x - p\,a = 0;$$

but $x^2 - a^2 = (x + a)\,.\,(x - a)$ [II. 5, cor. 1], and $p\,x - p\,a = p\,(x - a)$, which values being substituted, we have

$$(x + a)\,.\,(x - a) + p\,(x - a) = (x + a + p)\,.\,(x - a) = 0.$$

Now it is evident that the first member of this equation can only become equal to 0 by one of its factors becoming equal to 0; that is, either when $x - a = 0$, or when $x + a + p = 0$; and we thus see that there are two values of x, which will fulfil the conditions of the original equation (A); namely, either $x = a$, or $x = -\,a - p$. These values (a and $-\,a - p$) are termed the *roots* of the quadratic equation, and the symbols being perfectly general, we derive the following proposition:—

THEOREM 1. Every quadratic equation has two roots.

If x_1 and x_2 be put for the two values of x, and they be added together, we have

$$x_1 + x_2 = a - a - p = -\,p;$$

but p is the coefficient of the second term $p\,x$ in equation (B), therefore:—

THEOREM 2. The *sum* of the roots of a quadratic equation is equal to the coefficient of the second term with its sign changed.

Again, if the two roots be multiplied together, we have

$$x_1\,.\,x_2 = a\,.\,(-\,a - p) = -\,a^2 - a\,p;$$

but $q = -\,a^2 - ap$, and q is the third term of equation (B), therefore:—

THEOREM 3. The *product* of the roots of a quadratic equation is equal to the third term.

Now since $x_1 \,.\, x_2 = q$, $x_2 = \frac{q}{x_1}$;
substituting this value of x_2 in the equation $x_1 + x_2 = -p$, we have

$$x_1 + \frac{q}{x_1} = -p,$$

multiplying both sides by x_1,

$$x_1^2 + q = -p\,x_1,$$

transposing

$$x_1^2 + p\,x_1 = -q,$$

adding $\frac{p^2}{4}$ to both sides,

$$x_1^2 + p\,x_1 + \frac{p^2}{4} = \frac{p^2}{4} - q,$$

extracting the square root

$$x_1 + \frac{p}{2} = \sqrt{\frac{p^2}{4} - q}.$$

and again transposing

$$x_1 = -\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}. \quad \text{.......................} \quad (\text{C.})$$

Then from the equation $x_1 + x_2 = -p$, we have, by transposition, $x_2 = -p - x_1$, and substituting the value of x_1 in equation (C), we obtain

$$x_2 = -p + \frac{p}{2} - \sqrt{\frac{p^2}{4} - q},$$

and reducing

$$x_2 = -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}. \quad \text{.......................} \quad (\text{D.})$$

Now examining the equation $x^2 + b\,x = b^2$ already derived (page 73) from the algebraical expression of proposition xi., we immediately perceive that it is a quadratic equation, and may be reduced to the general form (B) by transposing b^2, when we have

$$x^2 + b\,x - b^2 = 0,$$

in which $p = b$, and $q = -b^2$. In order then to obtain the two roots of this equation, or the two values of x which will fulfil the conditions of the same, we have only to substitute the above values of p and q in the expressions (C) and (D); doing which, the first of these becomes

$$x_1 = -\frac{b}{2} + \sqrt{\frac{b^2}{4} + b^2},$$

which, by reduction, gives

$$x_1 = b\,\frac{\sqrt{5} - 1}{2},$$

a result identical with that already obtained at page 73 by a different process. And in a similar manner equation (D) becomes

$$x_2 = -\frac{b}{2} - \sqrt{\frac{b^2}{4} + b^2},$$

whence, by reduction,

$$x_2 = b\frac{\sqrt{5} \mp 1}{2},$$

also identical with the second value of x, previously obtained.

Now recollecting that $b = AB$, and $x = AH$, in the first case of the geometrical problem, when the point H is taken in the line AB itself,

$$AH_1 = AB\frac{\sqrt{5} - 1}{2},$$

and in the second case, when taken in the production of AB,

$$AH_2 = AB\frac{\sqrt{5} + 1}{2},$$

And this problem affords, geometrically, an illustration of the first theorem, enunciated at page 74, that every quadratic equation has two roots.

Now AH_2 (or AH in the second figure) is, from the construction, evidently equal to the sum of AB and AH_1 (or AH in the first figure), that is

$$AH_1 + AB = AH_2;$$

but AH_2 is negative, because measured in the opposite direction to AH (as already explained), therefore

$$AH_1 + AB = -AH_2,$$

and by transposition,

$$AH_1 + AH_2 = -AB,$$

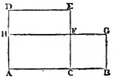
which is identical with the equation

$$x_1 + x_2 = -b,$$

and therefore a geometrical proof of the second theorem that the sum of the roots is equal to the coefficient of the second term.

Again, in the geometrical demonstration of the second case of the problem, it is shown that the rectangle (CG) under CK and CF is equal in area to the square (AD) on AB; but CF is equal to the value of AH in the first case (or AH_1), and CK is equal to its value in the second case (or AH_2); also AB is equal to b, and the square on AB to b^2, which is the third term; therefore the product of the two roots (AH_1 and AH_2) is equal to the third term (AB^2); and we hence derive a geometrical proof of the truth of the third theorem.

2. When a line is divided as in the eleventh proposition, it is said to be cut "in extreme and mean ratio." For the segment BH is to the segment AH as AH is to the whole line AB; and in such a proportion the product of the extremes equals the product of the means, or in this instance the rectangle under AB and BH is equal to the square on AH, which is proposition xi. itself.



COROLLARY 1. *If a line* (AB) *be cut in extreme and mean ratio* (in C), the greater segment will be cut in the same manner by taking on it a part equal to the less.

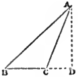
DEMONSTRATION. *On* AC *and* CB, *the two segments, construct the squares* ADEC *and* CFGB (*a*), *and produce* GF *to* H. Because BG is equal to CB (*b*), the rectangle AG is equal in area to the square AE (*c*), taking away from both the common rectangle AF, and the rectangle HE is equal in area to the square CG; but DE is equal to EC, therefore HE is the rectangle under EF and EC;

(*a*) I. 46.
(*b*) Constr.
(*c*) II. 11.

also EC is equal to the greater segment AC, and FC is equal to the less CB; therefore the rectangle under EC and EF is equal in area to the square on CF, and *the line* CE *is cut in extreme and mean ratio.*

PROPOSITION XII.

THEOREM.—*If a perpendicular be drawn from any of the acute angles of an obtuse-angled triangle* (ABC) *to the opposite side* (BC) *produced*, the square on the side (AB) subtending the obtuse angle is greater than the sum of the squares on the two sides (BC and CA), which contain the obtuse angle, by double the rectangle under the side (BC), which is produced, and the external segment (CD) between the obtuse angle and the perpendicular.



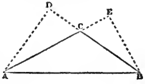
CONSTRUCTION. *Produce* BC, *and from the acute angle* A *draw* AD *perpendicular to* BC *produced* (*a*).

DEMONSTRATION. Because the straight line BD is divided into two parts in C, the square on BD is equal in area to the sum of the squares on BC and CD, together with double the rectangle under BC and CD (*b*); to each of these equals add the square on AD, and the sum of the squares on BD and AD is equal in area to the sum of the squares on BC, CD, and AD, together with double the rectangle under BC and CD. But because D is a right angle, the square on AB is equal in area to the sum of the squares on BD and AD (*c*), and the square on CA is equal in area to the sum of the squares on CD and AD (*c*); therefore the square on AB is equal in area to the sum of the squares on BC and CA together with double the rectangle under BC and CD; that is, *the square on* AB *is greater than the sum of the squares on* BC *and* CA *by double the rectangle under* BC *and* CD.

(*a*) I. 12.
(*b*) II. 4.
(*c*) I. 47.

COROLLARY. *If in any obtuse-angled triangle* (ABC) *the sides* (AC and CB) *which contain the obtuse angle be produced, and perpendiculars be drawn to the acute angles*, the rectangle under one of those sides (BC) and the produced part (CD) between the obtuse angle and the perpendicular, is equal in area to the rectangle under the other side (AC) and its produced part (CE).

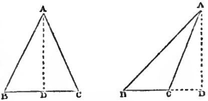
For it may be proved by the foregoing proposition that double the rectangle under AC and CE is also equal in area to the excess of the square on AB above the sum of the squares on BC and CA; therefore the rectangle under AC and CE is equal in area to the rectangle under BC and CD.



PROPOSITION XIII.

THEOREM.—*If in any triangle* (ABC) *a perpendicular be drawn to one of the sides* (BC) *which contains an acute angle, from the opposite angle*, the square on the side (AC) subtending that acute angle is less than the sum of the squares on the sides (AB and BC) which contain that angle, by double the rectangle under the side (BC) to which the perpendicular is drawn, and the segment (BD) between the perpendicular and the acute angle.

CONSTRUCTION. *From* A *draw* AD *perpendicular to* BC, *produced if necessary* (*a*).



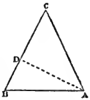
(*a*) I. 12.
(*b*) II. 7.
(*c*) I. 47.

DEMONSTRATION. Because when a straight line is divided, the sum of the squares on the whole line and one of the segments is equal in area to double the rectangle under the whole line and that segment together with the square on the other segment (*b*); therefore the sum of the squares on BD and CB is equal in area to double the rectangle under BD and CB, together with the square on CD; to each of these equals add the square on AD, and the sum of the squares on CB, BD, and AD is equal in area to the sum of the squares on CD and AD, together with double the rectangle under BD and CB. But, because the angles at D are right angles, the square on AB is equal in area to the sum of the squares on BD and AD (*c*), and the square on AC is equal in area to the sum of the squares on CD and AD (*c*); therefore the sum of the squares on AB and CB is equal in area to the square on AC together with double the rectangle under BD and CB; that is, *the square on* AC *is less than the sum of the squares on* AB *and* CB *by double the rectangle under* BD *and* CB.

SCHOLIUM. Euclid has separated this proposition into three cases, depending upon whether the perpendicular falls within or without the triangle, and employs the twelfth proposition to prove the second case. This division is not, however, necessary, as all the cases may be demonstrated from the seventh proposition, as is done above. By comparing the demonstrations of this and the preceding propositions, it will be seen how nearly identical they are, and they may be combined in one general proposition in the following terms:—"The difference between the square on one side of a triangle and the sum of the squares on the other two sides, is equal in area to double the rectangle under either of these two sides and the segment between the perpendicular on it from the opposite angle and the angle included by the sides."

Corollary 1. Theorem. *If in an isosceles triangle* (CBA) *a perpendicular be drawn from either angle of the base to the opposite side*, double the rectangle under that side (CB) and the segment (DB) between the perpendicular and the base is equal in area to the square on the base (AB).

For the sum of the squares on AB and CB is equal in area to the square on AC together with double the rectangle under CB and DB (*a*); but AC is equal to CB (*b*), and therefore the squares on them are equal (*c*); and taking away these equals, we have *the square on* AB *equal in area to double the rectangle under* CB *and* DB (*d*).

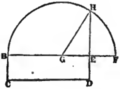


(*a*) II. 13.
(*b*) Hypoth.
(*c*) I. 46, cor. 1.
(*d*) Ax. 3.

PROPOSITION XIV.

Problem.—To construct a square that shall be equal in area to a given *rectilineal figure* (A).

Solution. *Construct a rectangle* BCDE *equal in area to the given rectilineal figure* (*a*); if the adjacent sides be equal, the problem is solved. *If not, produce either side* BE, *and make* EF *equal to the other side* ED (*b*); *bisect* BF *in* G (*c*), *and from the center* G, *at the distance* GF, *describe the semicircle* BHF; *produce* DE *to* H, *and join* GH; *then a square constructed on* EH *shall be equal in area to the given rectilineal figure.*



(*a*) I. 45.
(*b*) I. 3.
(*c*) I. 10.
(*d*) II. 5.
(*e*) I. Def. 13 and 16.
(*f*) I. 47.

Demonstration. Because the straight line BF is bisected in G, and also cut into two unequal parts in E, the rectangle under BE and EF, together with the square on GE, is equal in area to the square on GF (*d*), or of GH which is equal to GF (*e*). But the square on GH is equal to the square on GE together with the square on EH (*f*); and taking the square on GE away from both, the rectangle under BE and EF is equal in area to the square on EH; but EF is equal to DE, therefore the rectangle BD is equal in area to the square on EH; and therefore *the square on* EH *is equal in area to the rectilineal figure* A.

Scholium. From this proposition we derive the following theorem:—

If a perpendicular be drawn from any point in the circumference of a semicircular to the diameter, the square on the perpendicular is equal in area to the rectangle under the segments into which it divides the diameter.

THE
ELEMENTS OF EUCLID.

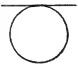
BOOK III.

DEFINITIONS.

1. Equal circles are those of which the diameters are equal, or those from the centers of which straight lines drawn to the circumferences are equal.

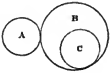
Scholium. This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centers coincide, the circles must likewise coincide, since the straight lines from the centers are equal.

2. A Tangent to a circle is a straight line which meets the circumference, *but being produced, does not cut the circle.*



3. Circles are said to touch one another when their circumferences meet, but do not cut one another.

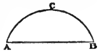
Scholium. They are said to touch *externally* when each circle is entirely without the other, as A and B; and the circumference of one circle is said to touch that of the other *internally* when one circle is entirely within the other, as B and C.



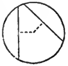
4. An Arc of a circle is any *portion* of the circumference.

5. A Chord is a straight line drawn within a circle, *whose extremities touch the circumference.*

Scholium. The portion of the circumference of a circle cut off by any chord is said to be the "arc *subtended* by that chord;" thus the arc ACB is subtended by the chord AB.



6. Straight lines (or chords) are said to be equally distant from the center of a circle when the perpendiculars drawn to them from the center are equal.

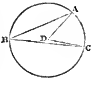


Scholium. And chords are said to be "farther from" or "nearer to" the center, according as the perpendiculars from them to the center are greater or less.

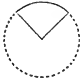
7. A SEGMENT of a circle is that portion of a circle *contained by a chord and its arc.*

8. An angle in a segment is the angle contained by the straight lines drawn from any point in the circumference of the segment to the extremities of its chord.

SCHOLIUM. An angle is said to *stand* upon the arc intercepted between the straight lines which contain the angle; thus the angle ABC is said to be the angle at the circumference standing upon the arc AC, and the angle ADC is said to be the angle at the center standing upon the same arc.



9. A SECTOR of a circle is that portion of a circle contained by two radii and the arc between them.



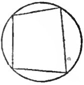
SCHOLIUM. When the radii are at right angles, the sector is termed a *quadrant*, being then equal to the fourth part of the whole circle.

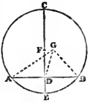
10. A segment of a circle is said to be *similar* to another segment, when an angle contained in one is equal to an angle contained in the other.

SCHOLIUM. The word "similar" is used in Geometry in a more strict sense than in ordinary conversation; in the latter it signifies mere resemblance, whereas in the former it is only used to express *perfect identity and sameness of form.* This definition, as given by Euclid, is an anticipation of the twenty-first proposition. We have above restricted the criterion of similarity to consist in one angle contained in each segment being equal.

11. A rectilineal figure is said to be *contained* by a circle, when its circumference passes through all its angular points.

SCHOLIUM. And the circle is said to be *described* about the figure.





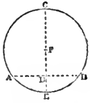
(a) Hypoth.
(b) I. 8.
(c) I. Def. 9.
(d) Ax. 11.

LEMMA. THEOREM. *If a straight line* (CE) *bisects the chord of a circle* (AB) *perpendicularly*, it contains the center of that circle.

DEMONSTRATION. For if not, let any point G without the line CE be the center of the circle. *Draw* GA, GD, *and* GB. Because in the triangles GAD and GBD, the side GA is equal to GB (*a*), DA equal to DB (*a*), and GD common to both, the angles GDA and GDB are equal (*b*), and therefore are right angles (*c*); but the angle CDB is also a right angle (*a*) therefore the angle GDB is equal to CDB (*d*), a part equal to the whole, which is absurd; therefore the point G is not the center of the circle ACB, and in like manner it may be shown that no other point without the line CE is the center, therefore *the line* CE *contains the center of the circle.*

SCHOLIUM. The above theorem has been introduced by way of lemma to the first proposition, to meet the objection which has been urged against Euclid's demonstration, because he therein assumes the position of the center previous to the perfect solution of the problem. Euclid's demonstration is further defective in two other respects; first, that it does not apply when the point G is assumed in the line CE itself, in which it may be without coinciding with the point F; and secondly, his demonstration is deficient, inasmuch as he only proves as much as is proved in the foregoing proposition, namely that the center must be contained in the line CE but he omits to *prove* in what part of that line it is situated.

PROPOSITION I.

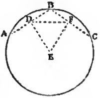


(a) I. 10.
(b) I. 11.
(c) Preceding lemma.
(d) Solution.
(e) I. Def. 15.

PROBLEM.—To find the center of a given *circle* (ACB).

SOLUTION. *Draw within the circle any straight line* AB, *bisect it in* D (*a*), *and from* D *draw* DC *perpendicular to* AB (*b*), *and produce it to* E; *bisect* CE *in* F (*a*), *and* F *is the center of the circle.*

DEMONSTRATION. Because the line CE bisects the chord AB of the circle ACB perpendicularly (*c*), it contains the center of the circle; and because it is bisected in F (*d*), FE is equal to FC, therefore F *is the center of the circle* (*e*).

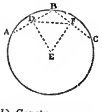


(a) I. 10 and 11.

THEOREM. *If three points* (A, B, and C) *are not in the same straight line*, a circle may be described whose circumference shall pass through them.

CONSTRUCTION. *Join* AB *and* BC, *bisect* AB *by the perpendicular* DE (*a*), *and* BC *by the perpendicular* FE (*a*).

DEMONSTRATION. The lines DE and FE will meet; for join DF, then the angles FDE and DFE are together less than the angles BDE and BFE, or than two right angles (*b*), and therefore they will meet in some point E (*c*); then every circle which passes through the points A and B has its center in the perpendicular DE (*d*), and every circle which passes through the points B and C has its center in the perpendicular FE (*d*); therefore *the circumference of the circle whose center is* E *passes through the three points* A, B, *and* C.

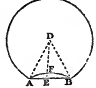


(*b*) Constr.
(*c*) Theor. attached to I. 29.
(*d*) Preceding lemma.

PROPOSITION II.

THEOREM.—*If any two points* (A and B) *be taken in the circumference of a circle,* the straight line which joins them falls within the circle.

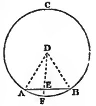
DEMONSTRATION. For, if it be possible, let the straight line AEB have a point E without the circle ACB. *Find the center* D (*a*), *and join* DA, DB, *and* DE, *cutting the circumference in* F. Because in the triangle ADB the side AD is equal to BD (*b*), the angle DAB is equal to the angle DBA (*c*); but the external angle DEA is greater than the internal angle DBA (*d*), and therefore greater than its equal DAB. Then in the triangle DAE, because the angle DEA is greater than the angle DAE, the opposite side DA is greater than DE (*e*), and therefore greater than DF; but DA and DF, being both radii of the same circle, are equal (*b*), which is absurd; therefore the point E is not without the circle, and in a similar manner it may be shown *that no point in the line* AB *is without the circle.*



(*a*) III. 1.
(*b*) I. Def. 15.
(*c*) I. 5.
(*d*) I. 16.
(*e*) I. 19.

SCHOLIUM. The foregoing proof being by the method "reductio ad absurdum," is necessarily indirect. By the admission of the axiom "That the extremity of a straight line less than the radius is within the circle,' the following direct proof may be given.

Take any point E, join DE, and produce it, if necessary, to meet the circumference in F; then it may be proved, as above, that DA is greater than DE, but DA and DF are equal (*a*), therefore DF is greater than DE, and E is by the above axiom within the circle.



(*a*) I. Def. 15.

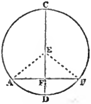
COROLLARY 1. A straight line cannot cut the circumference of a circle in more than two points.

COROLLARY 2. A straight line which touches the circumference of a circle meets it in only one point.

PROPOSITION III.

THEOREM.—(1.) *If a straight line* (CD) *drawn through the center of a circle bisect a straight line* (AB) *which does not pass through the center,* it is perpendicular to it. (2.) *And if it is perpendicular to it,* it bisects it.

CONSTRUCTION. *Find* E *the center of the circle* (*a*), *and join* EA *and* EB.



DEMONSTRATION. (1.) In the triangles EAF and EBF, because the side AE is equal to BE (*b*), the side AF to BF (*c*), and the side EF is common to both, the angle EFA is equal to EFB (*d*), and they are both right angles (*e*), therefore CD *is perpendicular to* AB.

(2.) Because in the triangle EAB the side AE is equal to EB (*b*), the angle EAF is equal to EBF (*f*). And because in the two triangles AEF and BEF, the angle EAF is equal to EBF, the angle EFA to EFB (*e*), and the side EF common to both, the side AF is equal to BF (*g*), and therefore *the line* CD *bisects the line* AB.

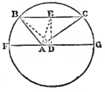
(*a*) III. 1.
(*b*) I. Def. 15.
(*c*) Hypoth.
(*d*) I. 8.
(*e*) I. Def. 9.
(*f*) I 5.
(*g*) I 26.

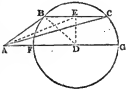
SCHOLIA. 1. The third proposition, as given by Euclid, consists of two distinct propositions, each the converse of the other, and are here separately distinguished. It has been explained in the Introduction that when a theorem has several hypotheses and one predicate (*i. e.* consequence), if another theorem be framed, having one of those hypotheses for its predicate and the former predicate for one of its hypotheses, the second theorem will be the *converse* of the first; and therefore, in fact, as many converse propositions might be framed as the first theorem contained hypotheses, although these converse propositions would not necessarily be all true. In this sense the two theorems above and the lemma given at the commencement of the Third Book are related, being mutually the converse of each other, which will be very evident by expressing them in the following manner:—

LEMMA.—*If a line bisects the chord of a circle; And is perpendicular to it;*	The line passes through the center of the circle.
PROP. III. (1).—*If a line passes through the center of a circle; And bisects a chord of the circle;*	The line is perpendicular to the chord.
PROP. III. (2).—*If a line passes through the center of a circle; And is perpendicular to a chord of the circle;*	The line bisects the chord.

2. It is necessary, in the enunciation of the third proposition, to state that the second line or chord AB shall not pass through the center of the circle; because if it did the two lines would intersect in the center of the circle, and would mutually bisect each other, whatever might be the angle in which they were inclined.

COROLLARY; THEOREM. *If from any point* (A) *in the diameter of a circle or its extension, lines be drawn to the ends of a parallel chord* (BC), the squares on those lines are together equal in area to the squares on the segments into which the point (A) divides the diameter.





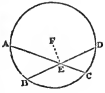
CONSTRUCTION. *From the center of the circle* D *draw* DE *perpendicular to the diameter* FG (*a*), *and join* AE *and* BD.

DEMONSTRATION. Because DE cuts BC perpendicularly, it bisects it (*b*); therefore in the triangle BAC the sum of the squares on AB and AC is equal in area to twice the square on BE together with twice the square on AE (*c*). But because ADE is a right angle, twice the square on AE is equal in area to twice the square on AD together with twice the square on DE (*d*); therefore the sum of the squares on AB and AC is equal in area to twice the square on BE, twice the square on DE, and twice the square on AD taken together. But because BED is a right angle, twice the square on BE together with twice the square on DE is equal in area to twice the square on BD (*d*), or its equal DG (*e*); therefore the sum of the squares on AB and AC is equal in area to twice the square on AD together with twice the square on DG. But because the line FG is bisected in D, and also cut unequally in A, the sum of the squares on AF and AG is equal in area to twice the square on AD together with twice the square on DG (*f*); therefore *the sum of the squares on* AB *and* AC *is equal in area to the sum of the squares on* AF *and* AG. When the point A is in the diameter produced (as in the second figure), the latter part of the proof is drawn from II. 10 instead of II. 9.

(*a*) I. 11.
(*b*) III. 3 (2).
(*c*) II. 9, cor. 1.
(*d*) I. 47.
(*e*) I. Def. 15.
(*f*) II. 9.

PROPOSITION IV.

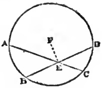
THEOREM.—*If in a circle* (ABCD) *two straight lines* (AC and BD) *cut one another, which do not both pass through the center,* they do not bisect each other.



DEMONSTRATION. If one of the lines pass through the center, it is evident that it cannot be bisected by the other which does not pass through the center. But if neither of the lines pass through the center, let it be possible that AC and BD bisect each other in E; *find the center of the circle* F (*a*), *and join*

(*a*) III. 1.

FE. Then because FE bisects AC, the angle FEA is a right angle (*b*), and because FE bisects BD, the angle FEB is also a right angle (*b*); therefore the angle FEA is equal to the angle FEB (*c*), the less equal to the greater, which is absurd; therefore *the lines* AC *and* BD *do not bisect each other.*

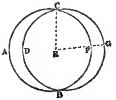


(*b*) III. 3 (1).
(*c*) Ax. 11.

PROPOSITION V.

Theorem.—*If the circumferences of two circles* (ABC and CDG) *cut one another* (in C and B), they have not the same center.

Demonstration. For, if it be possible, let E be the center of both circles. *Join* EC, *and draw any straight line* EG *cutting the circumferences of the two circles in* F *and* G. Then because E is the center of the circle ABC, CE is equal to EF (*a*); and because E is the center of the circle CDG, CE is equal to EG (*a*); therefore EF is equal to EG (*b*), the less equal to the greater, which is absurd; therefore E is not the center; and in the same manner it may be proved that *no other point is the center of both circles.*



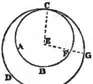
(*a*) I. Def. 15.
(*b*) Ax. 1.

Scholia. 1. This proposition applies equally when the circles are in contact as when they intersect, and may be otherwise expressed thus:—"*If two concentric circles have a point of the circumference common to both,* they coincide." The sixth proposition is therefore really included in the foregoing.

2. Euclid occasionally employs the word "circle," in the Elements, in two senses, namely, in one to denote the plain figure contained by the circumference, and in the other merely the circumference itself. In the following pages it is restricted to the first, which is its correct meaning according to the thirteenth definition in the First Book.

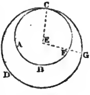
PROPOSITION VI.

Theorem.—*If the circumference of one circle* (ABC) *touch the circumference of another circle* (CDG) *internally* (in C), they have not the same center.



Demonstration. For, if it be possible, let E be the center of both circles. *Join* EC, *and*

draw any straight line EG, *cutting the circumferences of the two circles in* F *and* G. Then, because E is the center of the circle ABC, CE is equal to EF (*a*); and because E is the center of the circle CDG, CE is equal to EG (*a*); therefore EF is equal to EG (*b*), the less equal to the greater, which is absurd; therefore E is not the center; and in the same manner it may be proved that *no other point is the center of both circles.*



(*a*) I. Def. 15.
(*b*) Ax. 1.

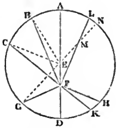
SCHOLIUM. The letters in the diagram to this proposition have been transposed, in order to show that the fifth and sixth propositions are identical; and the enunciation has been altered, because the two circumferences cannot *touch one another internally*, for one must be *external* to the other.

PROPOSITION VII.

THEOREM.—*If from any point* (F) *within a circle* (ABCD), *which is not the center, straight lines be drawn to the circumference,* [1] the greatest is that which passes through the center; [2] the remaining part of the diameter is the least; [3] that line which is nearer to the line (AD) passing through the center, is greater than one more remote; [4] and more than two straight lines cannot be drawn which shall be equal.

CONSTRUCTION. *Find the center of the circle* E (*a*), *and join* EB, EC, *and* EG.

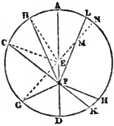
DEMONSTRATION. [1.] Because two sides of a triangle are greater than the third (*b*), BE and EF are greater than BF; but AE is equal to BE (*c*), therefore AE and EF, that is AF, *which passes through the center, is greater than* BF.



(*a*) III. 1.
(*b*) I. 20.
(*c*) I. Def. 15.
(*d*) I. 24.

[2.] Because two sides of a triangle are greater than the third (*b*), GF and FE are greater than GE; but GE is equal to ED (*c*), therefore GF and FE are greater than ED; and taking FE from both, GF *is greater than* FD, *the remaining part of the diameter.*

[3.] When the lines BF and CF are on the same side of the center; because in the triangles BEF and CEF, the side BE is equal to CE (*c*), the side FE common to both, and the angle BEF greater than the angle CEF, therefore *the side* BF *nearer to the diameter is greater than* CF (*d*), *the one more remote.*



If the lines CF and LF are on opposite sides of the center, *construct the angle* EFB *equal to the angle* EFL (*e*), then BF is equal to LF; for if not, let one of them LF be the greater, and cut off FM equal to BF, then because in the triangles BEF and MEF the side BF is equal to MF, the side EF common to both, and the angle EFB equal to EFM, the side BE is equal to EM (*f*); but BE is equal to EN (*c*), therefore EM is equal to EN (*g*), a part equal to the whole, which is absurd; therefore BF is equal to LF; but BF is greater than CF, and therefore LF, *which is nearer to the diameter, is greater than* CF, *the one more remote.*

(*c*) I. Def. 15.
(*e*) I. 23.
(*f*) I. 4.
(*g*) Ax. 1.

[4.] *Construct the angle* DFH *equal to* DFG (*e*), then FH is equal to GF, but no other line than FH can be drawn from F to the circumference, which shall be equal to GF; for, if it be possible, let FK be equal to GF, then FK is equal to FH (*g*), the nearer equal to the more remote, which is impossible; therefore *no more than two straight lines can be drawn from the same point, which shall be equal.*

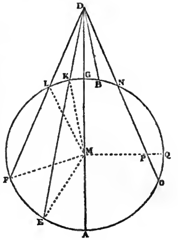
SCHOLIUM. Euclid's enunciation of this proposition really consists of four distinct theorems, which are all consequences following from the first assumed hypothesis; the following may be added, being proved in the demonstration of the third theorem above, but not there distinctly enunciated, namely,—

If from any point within a circle, which is not the center, straight lines be drawn to the circumference, those lines which form equal angles with the line passing through the center are equal.

PROPOSITION VIII.

THEOREM.—*If from any point* (D) *without a circle* (ABF) *straight lines be drawn to the circumference,* [1] of those which fall on the concave circumference, the greatest is that which passes through the center; [2] of the rest, that which is nearer to the line passing through the center is greater than the more remote; [3] but of those which fall on the convex circumference, the least is that which, if produced, would pass through the center; [4] of the rest, that which is nearer to the least is less than the more remote; [5] and more than two straight lines cannot be drawn, either to the concave or convex circumference, which shall be equal.

CONSTRUCTION. *Find the center of the circle* M (*a*), *and join* ME, MF, ML, *and* MK.



DEMONSTRATION. [1.] Because two sides of a triangle are greater than the third (*b*), DM and ME are greater than DE; but MA is equal to ME (*c*), therefore DM and MA, that is DA, *which passes through the center, is greater than* DE.

[2.] When the lines DE and DF are on the same side of the center; because in the triangles DME and DMF the side ME is equal to MF (*c*), the side DM common to both, and the anglo DME greater than the angle DMF, therefore *the side* DE *nearer to the line passing through the center is greater than* DF (*d*), *the one more remote.*

If the lines DF and DO are on opposite sides of the center, *construct the angle* EDM *equal to the angle* ODM (*e*); then DE is equal to DO; for if not, let one of them DO be the greater, and cut off DP equal to DE, then because in the triangles DEM and DPM the side DE is equal to DP, the sido DM common to both, and the angle EDM equal to PDM, the side ME is equal to MP (*f*); but ME is equal to MQ (*c*), therefore MP is equal to MQ (*g*), a part equal to the whole, which is absurd; therefore DE is equal to DO; but DE is greater than DF, and therefore DO, *which is nearer to the line passing through the center, is greater than* DF, *the one more remote.*

(*a*) III. 1.
(*b*) I. 20.
(*c*) I. Def. 15.
(*d*) I. 24.
(*e*) I. 23.
(*f*) I. 4.
(*g*) Ax. 1.

[3.] Because two sides of a triangle are greater than the third (*b*), DK and KM are greater than DM; but MK is equal to MG (*c*), therefore DK and MG are greater than DM; and taking MG from both, DK *is greater than* DG, *which, if produced, would pass through the center.*

[4.] When the lines DK and DL aro on the same side of the line passing through the center; because in the triangles DLM and DKM the side LM is equal to KM (*c*), the side DM common to both, and the angle LMD greater than KMD, the side DL *is greater than* DK, *which is the nearer to the line which, if produced, passes through the center.*

If the lines DL and DB are on opposite sides of the line passing through the center, *construct the angle* MDK *equal to the angle* MDB (*e*); then DK is equal to DB, but DL is greater than DK,

therefore DL *is greater than* DB, *which is the nearer to the line which, if produced, passes through the center.*

[5.] *Construct the angle* MDB *equal to the angle* MDK (*e*), then DB is equal to DK, but no other line but DB can be drawn from D to the circumference which shall be equal to DK; for, if it be possible, let DN be equal to DK, then DB is equal to DN (*g*), the nearer equal to the more remote, which is impossible; therefore *no more than two straight lines can be drawn from the same point which shall be equal.*

SCHOLIUM. The seventh and eighth propositions may be regarded simply as particular cases of one more general theorem, in the former the point being taken within the circle, and in the latter without it; the following theorem may be added to those enunciated above, namely,—

If from any point without a circle straight lines be drawn to the circumference, those lines which form equal angles with the line passing through the center are equal.

PROPOSITION IX.

THEOREM.—*If a point be taken within a circle, from which more than two equal straight lines can be drawn to the circumference*, that point is the center of the circle.

DEMONSTRATION. For if it were a point different from the center, only two equal straight lines could be drawn from it to the circumference (*a*), therefore *a point from which three equal straight lines can be drawn can be no other than the center of the circle.*

(*a*) III. 7.

SCHOLIUM. This proposition, as pointed out by Mr. De Morgan, is only a logical equivalent of Proposition vii. (4), for it is there shown that—

Only two equal straight lines can be drawn to any point which is not the center;

But by the hypothesis,—

From a certain point three equal straight lines can be drawn;

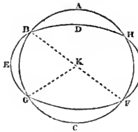
Therefore—

That point is not a point which is not the center; *i. e. it is a point which is the center.*

PROPOSITION X.

THEOREM.—*If two lines* (FAB and DEF) *be the circumferences of two circles,* they cannot cut one another in more than two points.

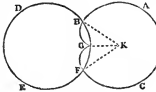
DEMONSTRATION. For, if it be possible, let the circumference FAB cut the circumference DEF in the three points B, G, and F.



Find the center K *of the circle* ABC (*a*), *and join* KB, KG, and KF; then these lines are equal (*b*); and because within the circle DEF, a point K is taken from which three equal straight lines are drawn to the circumference, that point is the center of the circle DEF (*c*), but it is also the center of the circle ABC (*d*); therefore the same point K is the center of two circles whose circumferences cut each other, which is impossible (*e*); therefore *one circumference of a circle cannot cut another in more than two points.*

(*a*) III. 1.
(*b*) I. Def. 15.
(*c*) III. 9.
(*d*) Constr.
(*e*) III. 5.

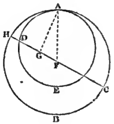
Scholium. The above mode of demonstrating this proposition cannot be applied when the centers of the circles are without each other. In such case the following proof may be given:—



Let it be possible that the two circumferences cut one another in the three points B, G, and F. *Find the center* K *of the circle* ABC (*a*), *and join* KB, KG, *and* KF; then these lines are equal (*b*); but as the circumferences of the circles ABC and DEF cut one another, they have not the same center (*c*), therefore K is not the center of the circle DEF; and because no more than two equal straight lines can be drawn to the circumference from a point which is not the center, therefore KB, KG, and KF are not equal (*d*); but it has been proved that they are equal, which is absurd; therefore *one circumference of a circle cannot cut another in more than two points.*

(*a*) III. 1.
(*b*) I. Def. 15.
(*c*) III. 5.
(*d*) VII. 8 (5).

PROPOSITION XI.

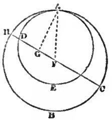


Theorem.—*If the circumference of one circle* (ADE) *touch the circumference of another circle* (ABC) *internally in any point* (A), the straight line joining their centers, being produced, shall pass through that point.

Demonstration. *Find the center* F *of the circle* ABC (*a*), *and the center* G *of the circle* ADE (*a*); *draw the straight line* HGFC, *passing through* F *and* G, *produce it to meet*

(*a*) III. 1.

the circumferences, and, if it be possible, let it not pass through the point of contact A; *join* AF *and* AG. Then in the triangle AGF, AG and GF are greater than AF (*b*); but AF and HF are equal (*c*); therefore AG and GF are greater than HF; take away the common part GF, and AG is greater than HG. But AG is equal to DG (*c*); therefore DG is greater than HG, the part greater than the whole, which is impossible; therefore *the straight line passing through the centers* F *and* G, *being produced, also passes through the point of contact* A.



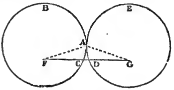
(*b*) I. 20.
(*c*) I. Def. 15.

SCHOLIUM. In Euclid's enunciation of this proposition he speaks of "*the* point of contact," thereby tacitly assuming that there is only one; which assumption, although correct, is the subject of after proof in the thirteenth proposition, and ought not therefore to have been anticipated. The enunciation as given above is so expressed as to remove this objection.

PROPOSITION XII.

THEOREM.—*If the circumferences of two circles* (ABC and ADE) *touch each other externally in any point* (A), the straight line joining their centers shall pass through that point.

DEMONSTRATION. *Find the center* F *of the circle* ABC (*a*), *and the center* G *of the circle* ADE (*a*); *join* F *and* G *with the straight line* FG, and, if it be possible, let it not pass through the point of contact A; *join* AF *and* AG. Then in the triangle AGF, AF and AG are greater than FG (*b*); but AF is equal to FC (*c*), and AG is equal to DG (*c*), therefore FC and DG are greater than FG, the part greater than the whole, which is absurd; therefore *the straight line joining the centers* F *and* G *does pass through the point of contact* A.

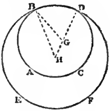


(*a*) III. 1.
(*b*) I. 20.
(*c*) I. Def. 15.

SCHOLIUM. From the two preceding propositions it appears that if the circumferences of two circles touch each other, the distance between their centers is equal to the sum or difference of the radii of those circles according as the circumference of one circle touches that of the other externally or internally.

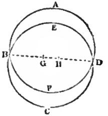
PROPOSITION XIII.

THEOREM.—*If the circumference of one circle* (EBF) *touch the circumference of another circle* (ABC) *either internally or externally*, there can only be one point of contact.

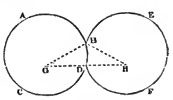


(*a*) III. 1.
(*b*) III. 11.
(*c*) I. 20.
(*d*) I. Def. 15.

DEMONSTRATION. For, if it be possible, let the circumference of the circle EBF cut that of the circle ABC internally in the two points B and D; *then find the centers* H *and* G *of the two circles* (*a*), *draw the straight line* GH *joining them, and produce it until it passes through the point of contact* D (*b*); *join* BH *and* BG. Then in the triangle BGH, BG and GH are greater than BH (*c*), but BH and DH are equal (*d*); therefore BG and GH are greater than DH; take from each the common line GH, and BG is greater than GD; but they are also equal (*d*), which is absurd; therefore *the circumferences of the two circles cannot touch in more than one point.*



But if the straight line drawn through the centers G and H passes through both points of contact B and D, it must be bisected in G, because BG is equal to GD (*d*), and also in H, because BH is equal to HD (*d*), which is absurd; therefore *the circumferences of the two circles cannot touch in more than one point.*

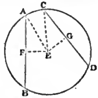


(*e*) III. 12.
(*f*) Ax. 1.

Next, if it be possible, let the two circumferences ABC and EBF touch each other externally in the two points B and D; *find the centers* G *and* H (*a*), *draw the straight line joining them and passing through one of the points of contact* D (*f*), *and join* BG *and* BH. Then in the triangle BGH, BG and BH are greater than GH (*c*); but BG is equal to DG (*d*), and BH is equal to DH (*d*), therefore BG and BH are equal to GH (*f*); but BG and BH are also greater than GH, which is absurd; therefore *the circumferences of the two circles cannot touch in more than one point.*

PROPOSITION XIV.

THEOREM.—[1] *If two straight lines in a circle* (AB and CD) *are equal,* they are equally distant from the center; [2] *and if straight lines* (AB and CD) *are equally distant from the center* they are equal.



CONSTRUCTION. *Find the center* E *of the circle* ABDC (*a*), *and from it draw* EF *and* EG *perpendicular to* AB *and* CD (*b*); *join* AE *and* CE.

(*a*) III. 1.
(*b*) I. 12.
(*c*) Constr.
(*d*) III. 3 (2).
(*e*) Hypoth.
(*f*) I. Def. 15.
(*g*) I. 46, cor. 1.
(*h*) I. 47.
(*i*) I. 46, cor. 2.
(*k*) Ax. 3.
(*l*) Ax. 6.

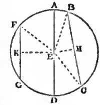
DEMONSTRATION. [1.] Because the straight line EF drawn through the center of the circle is perpendicular to the straight line AB (*c*) which does not pass through the center, it bisects it (*d*), therefore AF is equal to FB, and AB is double of AF; and for the same reason CD is double of CG. But AB is equal to CD (*e*), therefore AF is equal to CG. Then because AE is equal to CE (*f*), the square on AE is equal to the square on CE (*g*); but because AFE is a right angle, the squares on AF and FE are together equal in area to the square on AE (*h*); and because CGE is a right angle, the squares on CG and GE are together equal in area to the square on CE (*h*); therefore the squares on AF and FE are equal to the squares on CG and GE; but because AF is equal to CG, the square on AF is equal to the square on CG; therefore the remaining squares FE and GE are equal, and therefore the straight lines FE and GE are equal (*i*); but straight lines are said to be equally distant from the center of a circle when the perpendiculars drawn to them from the center are equal; therefore AB *and* CD *are equally distant from the center of the circle.*

[2.] Because AE is equal to CE (*f*), the square on AE is equal to the square on CE (*g*); and because FE is equal to EG (*e*), the square on FE is equal to the square on EG (*g*); but because AFE is a right angle, the squares on AF and FE are together equal in area to the square on AE (*h*); and because CGE is a right angle, the squares on CG and EG are together equal in area to the square on CE (*h*); therefore the squares on AF and FE are together equal in area to the squares on CG and EG. Take away from these equals the equal squares on FE and EG, and the remaining squares on AG and CG are equal (*k*), and therefore the lines AF and CG themselves are equal (*i*); but because the lines FE and EG bisect AB and CD (*d*), AF and CG are halves of AB and CD; and since AF and CG are equal, *their doubles* AB *and* CD *are also equal* (*l*).

SCHOLIUM. It will be seen that the foregoing consists of two propositions the *converse* of each other. They are here separately distinguished and demonstrated.

PROPOSITION XV.

THEOREM.—*If straight lines be drawn in a circle, of which one* (AD) *passes through its center,* [1] that line is the greatest; [2] and of all others, that which is nearer to the center (BC) is greater than (FG) the more remote; [3] and the greater (BC) is nearer to the center than the less (FG).



CONSTRUCTION. *Find the center* E *of the circle* ABCD (*a*), *and from it draw* EH *and* EK *perpendicular to* BC *and* FG (*b*); *join* BE, CE, *and* FE.

DEMONSTRATION. [1.] Because AE is equal to BE, and DE to CE (*c*), AD is equal to BE and CE; but in the triangle BEC, BE and CE are greater than BC (*d*), therefore AD *is greater than* BC.

(*a*) III. 1.
(*b*) I. 12.
(*c*) I. Def. 15.
(*d*) I. 20.
(*e*) III. Def. 6.

[2.] And because BC is nearer to the center than FG, EH is less than EK (*e*); but, as was demonstrated in the preceding proposition, BC is double of BH, and FG double of FK, and the squares on EH, HB are together equal in area to the squares on EK, FK, of which the square on EH is less than the square on EK, because EH is less than EK; therefore the square on BH is greater than the square on FK, and the straight line BH greater than FK; and therefore BC *is greater than* FG.

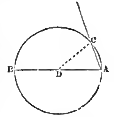
[3.] Because BC is greater than FG, BH likewise is greater than FK; but the squares on BH and EH are together equal in area to the squares on FK and EK, of which the square on BH is greater than the square on FK, because BH is greater than FK; therefore the square on EH is less than the square on EK, and the straight line EH less than EK, and therefore *the greater line* BC *is nearer to the center than the less* FG.

SCHOLIUM. In the above proposition we have followed Simson's demonstration, which differs from Euclid's, for the purpose of more correctly proving the converse of the second theorem. The proposition, as it stands above, consists of three distinct theorems, the hypothesis in each being the same.

PROPOSITION XVI.

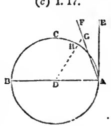
THEOREM.—[1] *If a straight line be drawn from the extremity of the diameter* (AB) *of a circle, perpendicular to the same,* it will fall without the circle; [2] *and if any straight line be drawn from a point between that perpendicular and the circle to the point of contact,* it will cut the circumference of the circle.

DEMONSTRATION. For, if it be possible, let CA be perpendicular to BA, and meet the circumference of the circle again in C, and *join* CD. Because in the triangle CDA the side CD is equal to AD, therefore the angles ACD and A are equal (*a*); but A is a right angle (*b*), therefore ACD and A are two right angles, which is impossible (*c*); therefore *the straight line drawn from* A *perpendicular to* AB *does not fall within the circle.*



(*a*) I. 5.
(*b*) Hypoth.
(*c*) I. 17.

[2.] Let EA be perpendicular to AB, and, if it be possible, let FA be a line drawn from the point F between it and the circle, which does not cut the circumference of the circle. *Draw* DG *perpendicular to* FA (*d*). Then in the triangle DAG, because the angle GAD is less than DGA, the side DG is less than DA (*e*). But DH and DA are equal (*f*), therefore DG is less than DH, the whole less than a part, which is absurd; therefore *no straight line can be drawn from a point between the perpendicular and the circle, which shall not cut the circumference of the circle.*



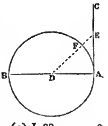
(*d*) I. 12.
(*e*) I. 19.
(*f*) I. Def. 15.

COROLLARY. Hence it is manifest that the straight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circumference of the circle, and that it touches it only in one point; and further, that only one straight line can touch the same point in the circumference of a circle.

SCHOLIA. 1. In the Greek text and the Latin translations the following is added to the enunciation of this proposition, namely, "That the angle between the diameter and circumference is greater than any rectilineal angle; and that the angle between the circumference and tangent is less than any rectilineal angle." This has been omitted because it involves the comparison of quantities which have been regarded by many as not homogeneous, and is in itself of no use.

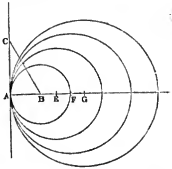
2. The foregoing proposition consists of two perfectly distinct theorems; the first may be directly proved as follows:—

Let AC be perpendicular to AB; take any point E in AC, and join DE. Then because DAE is a right angle, the angles EDA and DEA are together equal to a right angle (*a*), therefore the angle DEA is less than a right angle; and because in the triangle DAE the angle DEA is less than DAE, the side DA is less than DE (*b*); but DF is equal to DA (*c*), therefore DF is less than DE, and the point E is without the circle, and therefore *the whole line* AC *is without the circle.*



(*a*) I. 32. B., cor. 2.
(*b*) I. 19.
(*c*) I. Def. 15.

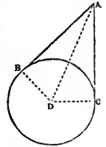
3. By means of this proposition it may be shown that any linear magnitude is capable of being infinitely divided: for let CA be perpendicular to AB, take any point C and join CB; then if AB be produced, any number of points, as E, F, G, &c., may be taken in the produced part, from each of which a circle may be described through the point A; and since two circles can only touch in one point, none of those circles shall meet again, but will cut the line CB each in a different point, and the line is therefore infinitely divisible.



4. From this proposition it is evident that the straight line which makes an acute angle with the diameter at A, however great, must meet the circle again.

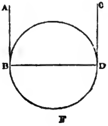
COROLLARY 1. *If tangents* (AB and AC) *are drawn from the same point to a circle,* they are equal to one another.

For in the triangles DAB and DAC the side BD is equal to DC (*a*), the side AD common to both, and the angle B equal to the angle C, being both right angles (*b*), therefore the remaining sides are equal (*c*), AB to AC.



(*a*) I. Def. 15.
(*b*) Ax. 11.
(*c*) I. 4.

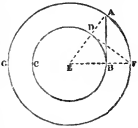
COROLLARY 2. *If tangents* (AB and CD) *are at the extremities of the same diameter,* they are parallel to one another.



PROPOSITION XVII.

PROBLEM.—From a given *point, either without a given circle or in its circumference,* to draw a straight line touching the circumference.

SOLUTION. Let A be the given point without the circle BCD. *Find the center* E *of the circle* (*a*), *and join* AE; *and from the center* E, *with the radius* EA, *describe the circle* AFG; *from the point* D *draw* DF *perpendicular to* EA (*b*); *and join* EF *and* AB. *The line* AB *touches the circle* BCD.



DEMONSTRATION. Because in the triangles AEB and FED the side AE is equal to FE (*c*), the side BE to DE, and the angle AEF common to both, the angles ABE and FDE at the base are equal (*d*); but FDE is a right angle (*e*), therefore ABE is also a right angle, and AB *touches the circumference of the circle* (*f*).

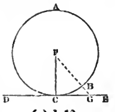
(*a*) III. 1.
(*b*) I. 11.
(*c*) I. Def. 15.
(*d*) I. 4.
(*e*) Solution.
(*f*) III. 16, cor.

But if the given point be in the circumference of the circle, as the point D, draw DE to the center E, and DF perpendicular to it; then DF *touches the circumference of the circle* (*f*).

SCHOLIUM. When the given point is without the circle, it is evident that two tangents can be drawn from it to the circle. Further, by the first corollary appended to the preceding proposition, all tangents drawn from any point to a circle are equal, and by the eighth proposition only two equal lines can be so drawn, therefore *only* two tangents can be drawn from a point without a circle to that circle.

PROPOSITION XVIII.

THEOREM.—*If a straight line* (DE) *touches the circumference of a circle,* the straight line (FC) drawn from the center to the point of contact shall be perpendicular to the line touching the circle.



(*a*) I. 12.

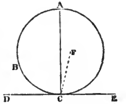
DEMONSTRATION. For if it be not, from the point F draw FBG perpendicular to DE (*a*). Then in the triangle FCG, because FGC is a right angle, FCG is less than a

right angle (*b*), and therefore the side FC is greater than FG (*c*); but FB is equal to FC (*d*); therefore FB is greater than FG, a part greater than the whole, which is absurd; therefore FG is not perpendicular to DE; and in the same manner it may be shown *that no other line than* FC *is perpendicular to* DE.

(*b*) I. 17.
(*c*) I. 19.
(*d*) I. Def. 15.

PROPOSITION XIX.

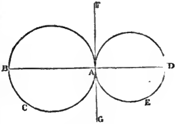
THEOREM.—*If a straight line* (DE) *touches the circumference of a circle, and a straight line* (CA) *be drawn perpendicular to it from the point of contact,* the center of the circle shall be in that line.



DEMONSTRATION. For, if it be possible, let the center of the circle F be without the line CA, and *join* CF. Because DE touches the circle ABC, and FC is drawn from the center to the point of contact, FC is perpendicular to DE (*a*), and the angle FCE is a right angle; but ACE is also a right angle (*b*); therefore the angle FCE is equal to ACE (*c*), the less to the greater, which is impossible; therefore F is not the center of the circle; and in the same manner it may be shown *that no point which is not in* CA *is the center of the circle.*

(*a*) III. 18.
(*b*) Hypoth.
(*c*) Ax. 11.

COROLLARY. *If the circumferences of two circles* (ABC and ADE) *touch each other in any point* (A), they have the same tangent at the point of contact.



CONSTRUCTION. *Through the point of contact* A *draw* FG, *touching the circle* ABC (*a*), *and through the same point draw* BAD *perpendicular to* FG (*b*).

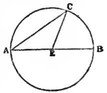
DEMONSTRATION. Because FG touches the circle ABC, and BA is perpendicular to it, the center of the circle is in BA (*c*); and because the straight line BD passes through the center of one of the circles and the point of contact, it also passes through the center of the other circle ADE (*d*), and therefore AD is a diameter of that circle; and because FG is drawn at right angles to the diameter AD (*e*), it touches the circumference of the circle ADE (*f*). Therefore *the line* FG *is a common tangent to the circles* ABC *and* ADE.

(*a*) III. 17.
(*b*) I. 11.
(*c*) III. 19.
(*d*) III. 12.
(*e*) Hypoth.
(*f*) III. 16, cor.

PROPOSITION XX.

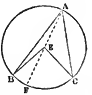
THEOREM.—*If an angle* (BEC) *at the center of a circle have the same part of the circumference for its base as an angle* (BAC) *at the circumference,* the former angle is double the latter.

DEMONSTRATION. [1.] Let one side of the angle at the circumference pass through the center of the circle. Because in the triangle AEC the side AE is equal to CE, the angle A is equal to C (*a*), therefore the angles A and C are together double the angle A; but the external angle CEB is equal to the internal and opposite angles A and C (*b*), therefore CEB, *the angle at the center, is double* A, *the angle at the circumference.*



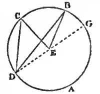
(*a*) I. 5.
(*b*) I. 32 A.

[2.] Let the center of the circle be within the angle at the circumference. *Join* AE *and produce it to* F. Then the angle BEF is double the angle BAF (*c*), and the angle FEC is double FAC (*c*); but if two magnitudes be double of two others, each to each, the sum of the first two is double the sum of the other two (*d*); therefore *the sum of the angles* BEF *and* FEC, *or the whole angle* BEC, *is double the sum of the angles* BAF *and* FAC, *or the whole angle* BAC.



(*c*) Preceding part.
(*d*) Scholium 2.

[3.] Let the center of the circle be without the angle at the circumference. *Join* DE *and produce it to* G. Then the angle GEB is double the angle GDB (*c*), and the angle GEC is double the angle GDC (*c*); but if two magnitudes be double of two others, each to each, the difference of the first two is double the difference of the other two (*d*); therefore *the difference of the angles* GEC *and* GEB, *or the angle* BEC, *is double the difference of the angles* GDC *and* GDB, *or the angle* BDC.



SCHOLIA. 1. Euclid has only given the last two of the preceding cases. The whole three are, however, here given to render the proposition complete.

2. In the second and third cases (at *d*) Euclid assumes two propositions which are really two cases of the first and fifth propositions of the Fifth Book, and which therefore ought to have been demonstrated. The propositions assumed have been inserted above, and their proof is as under.

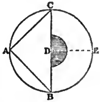
Let A *and* B, C *and* D, *be four magnitudes, such that* A *is double* C, *and* B *is double* D; then A added to B is double C added to D.

For since $A = C + C$, and $B = D + D$, adding equals together $A + B = C + C + D + D = (C + D) + (C + D) = 2(C + D)$.

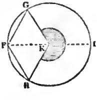
Again, *let* A *and* B, C *and* D, *be four magnitudes, such that* A *is double and* B *is double* D, *and let* A *be greater than* B; then the difference between A and B is double the difference between C and D.

For since A = C + C, and B = D + D, taking equals from equals, A — B = C + C — D — D = (C — D) + (C — D) = 2 (C — D).

3. It is here of importance to remember the extended meaning given to the term "angle" in the scholium attached to its definition (page 2), and to point out that the twentieth proposition holds true when the term "angle" is so understood. Thus when the angle at the center becomes equal to two right angles, as the [shaded] angle CDB, the angle at the circumference CAB may be shown to be a right angle, it being the angle in a semicircle (*a*); and when the angle at the center becomes still further increased, so as to exceed two right angles, as the [shaded] angle GKH, the angle at the circumference GFH may be readily shown to be equal to its half by drawing the line FI and following the demonstration of the second case above.



(*a*) III. 31.

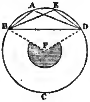


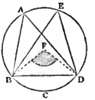
PROPOSITION XXI.

THEOREM.—*If angles* (A and E) *are in the same segment* (BAED) *of a circle*, they are equal to one another.

CONSTRUCTION. *Find the center* F *of the circle* (*a*), *and join* FB *and* FD.

DEMONSTRATION. Because the [shaded] angle BFD is at the center, and the angle A at the circumference, and they have the same part of the circumference BCD for their base, the [shaded] angle BFD is double the angle A (*b*); and for the same reason the [shaded] angle BFD is double the angle E; therefore *the angle* A *is equal to the angle* E.

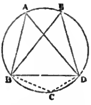




(*a*) III. 1.
(*b*) III. 20.

SCHOLIA. 1. Simson divides this proposition into two cases, namely, when the segment containing the angles is *less* than a semicircle, and when *greater*; and gives a separate demonstration to each. By taking the term "angle," however, in the extended sense explained in the scholia to the preceding proposition, this distinction becomes unnecessary, the same demonstration applying in both cases.

2. This proposition may be demonstrated in a different manner by means of the proposition which immediately follows it.

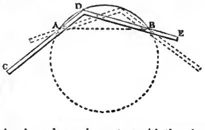


(a) III. 22.

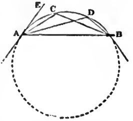
In the arc BCD take any point C, and join BC and DC. Then because the quadrilateral ABCD is contained in the circle, its opposite angles A and C are together equal to two right angles (a); and because the quadrilateral EBCD is contained in the circle, its opposite angles E and C are together equal to two right angles; therefore the angles A and C are together equal to E and C. From these equals take the common angle C, and *the angle* A *is equal to the angle* E.

3. From this proposition we derive the following theorem; namely, That the *locus* of the vertices of all triangles, standing upon the same base and having the same vertical angle, is a circular arc.

4. A practical method of describing circular arcs of large radius is derived from the preceding proposition; it consists in fixing a pin at each extremity A and B of the required arc, taking two rulers or straight-edges CD and DE, and joining them at a certain angle (depending upon the radius of the circle to be described) at D. If a tracer be fixed at the point D, and the rulers moved, keeping their edges always in contact with the pins A and B, the line described by it will be the circular arc required. An instrument has been contrived for this purpose, termed the *cyclograph;* it consists of two rulers connected by a joint, so as to admit of being set to any required angle.



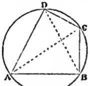
5. A method for setting out any portion of a circular arc of large radius on the ground, as, for example, the course of a railway, canal, or road, has also been proposed, which is founded upon this proposition. It consists in placing a theodolite at each extremity of the required arc, as A and B, one as A in the direction of the tangent AE, the other B in that of the chord AB. If each instrument be now moved in the same direction through an equal angle, as, for instance, A through the angle EAC, and B through the equal angle ABC, the point of intersection C of their axes will be a point in the circular arc; and by repeating the operation, and making the successive angles sufficiently small, any number of points may be found. If the instruments, having been fixed in the positions first supposed, were made to revolve in the same direction with an equal and uniform angular velocity, the point of intersection of their axes would describe a circular arc.



PROPOSITION XXII.

THEOREM.—*If a quadrilateral figure* (ABCD) *is contained within a circle,* its opposite angles are together equal to two right angles (ABC and ADC, or BAD and BCD).

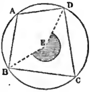
CONSTRUCTION. *Join* AC *and* BD.



DEMONSTRATION. In the triangle CAB the three angles ABC, BCA, and CAB are equal to two right angles (*a*); but the angle CAB is equal to the angle BDC, being both in the same segment BADC (*b*); and the angle ADB is equal to the angle BCA, being both in the same segment ADCB; therefore the whole angle ADC is equal to the angles CAB and BCA; to each of these equals add the angle ABC, and the angles ABC, BCA, and CAB are equal to the angles ADC and ABC; but the angles ABC, BCA, and CAB are equal to two right angles; therefore *the angles* ADC *and* ABC *are together equal to two right angles.* And in the same manner it may be proved that *the angles* BAD *and* BCD *are together equal to two right angles.*

(*a*) I. 32 B.
(*b*) III. 21.

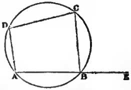
SCHOLIA. 1. Another and easier demonstration may be given of this proposition, derived from the twentieth proposition, in the following manner:—This proposition being thus proved without reference to the twenty-first, might be placed before it, and the twenty-first then proved from it in the manner shown in the scholia to that proposition.



Find the center E *of the circle* (*a*), *and join* BE *and* DE. The angle BED at the center is double the angle BCD at the circumference (*b*), and the [shaded] angle BED is double the angle BAD (*b*); therefore both the angles at the center E are together equal to double the angles BAD and BCD; but both the angles at the center are together equal to four right angles (*c*); therefore *the angles* BAD *and* BCD *are together equal to two right angles.* And in the same manner it may be shown that *the angles* ABC *and* ADC *are together equal to two right angles.*

(*a*) III. 1.
(*b*) III. 20.
(*c*) I. 13, cor. 3

COROLLARY 1. THEOREM. *If one side* (AB) *of a quadrilateral figure* (ABCD) *contained within a circle be produced,* the external angle (CBE) is equal to the angle (ADC) opposite to the internal adjacent angle.



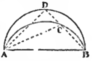
For the opposite angles ADC and ABC are together equal to two right angles (*a*), and the angles ABC and CBE are also equal to two right angles (*b*); therefore the angles ADC and ABC are together equal to the angles ABC and CBE, and taking from each the common angle ABC, *the angle* ADC *is equal to the angle* CBE (*c*).

(*a*) III. 22.
(*b*) I. 13.
(*c*) Ax. 1.

2. The *converse* of the twenty-second proposition is also true; namely, *If a quadrilateral has its opposite angles together equal to two right angles*, a circle may be described about it.

PROPOSITION XXIII.

THEOREM.—*If two segments of circles* (ACB and ADB) *are upon the same straight line* (AB) *and upon the same side of it,* they cannot be similar without coinciding with one another.

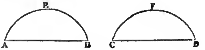


DEMONSTRATION. For, if it be possible, let the segments ACB and ADB be similar without coinciding. Then because the circumferences of the circles ACB and ADB cut one another in the two points A and B, they cannot cut in any other point (*a*); one of the segments must therefore fall within the other; then in the interior segment ACB take any point C, join BC and produce it to the exterior segment in D, and join CA and DA. Then because the segments ACB and ADB are similar, the angle ACB is equal to the angle ADB (*b*), the external to the internal, which is impossible (*c*). Therefore *the segments* ACB *and* ADB *cannot be similar without coinciding with one another.*

(*a*) III. 10.
(*b*) III. Def. 10.
(*c*) I. 16.

PROPOSITION XXIV.

THEOREM.—*If two segments of circles* (AEB and CFD) *are similar and upon equal straight lines* (AB and CD), they are equal to one another and have equal arcs.



DEMONSTRATION. For if the segment AEB be applied to the segment CFD, so that the point A be on C, and the straight line AB upon CD, the point B shall coincide with the point D, because AB is equal to CD; therefore the straight line AB coinciding with CD, *the segment* AEB *must coincide with segment* CFD (*a*), *and therefore is equal to it* (*b*); *and the arc* AEB *must coincide with the arc* CFD, *and be equal to it.*

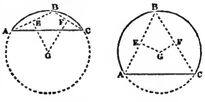
(*a*) III. 23.
(*b*) Ax. 8.

SCHOLIA. 1. This proposition is proved by the method of *superposition.* The equality of the arcs is not stated by Euclid.

2. Since if the circumferences of two circles coincide in more than two points, they must coincide in every part, it follows that similar segments on equal chords are parts of equal circles.

PROPOSITION XXV.

PROBLEM.—*A segment of a circle* (ABC) being given, to describe the circle of which it is the segment.



SOLUTION. From any point B draw two straight lines BA and BC; bisect them in E and F (*a*), and through E and F draw EG and FG perpendicular to AB and BC (*b*); the intersection G of the perpendiculars is the center of the circle.

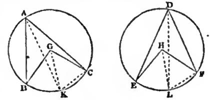
(*a*) I. 10.
(*b*) I. 11.
(*c*) Lemma preceding III. 1.

DEMONSTRATION. Because the straight line EG bisects the chord AB perpendicularly, it passes through the center (*c*), and because the line FG bisects the chord BC perpendicularly, it also passes through the center (*c*); therefore *the center must be in* G, *the intersection of* EG *and* FG.

SCHOLIUM. The above demonstration is substituted for the tedious one given by Euclid, who divides this proposition into three cases, namely, when the segment is less than, equal to, or greater than a semicircle.

PROPOSITION XXVI.

THEOREM.—*If equal angles* (BAC and EDF, or BGC and EHF) *are in equal circles* (ABC and DEF), *or in the same circle*, they shall stand upon equal parts of the circumference, whether they be at the center or the circumference.

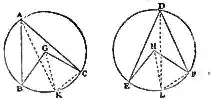


CONSTRUCTION. *Bisect the equal angles* BGC *and* EHF *by the lines* GK *and* HL (*a*), *cutting the circumferences in* K *and* L; *join* KC, LF, AK, *and* DL.

(*a*) I. 9.

DEMONSTRATION. Because the circles ABC and DEF are equal, their radii are equal; therefore in the triangles GKC and HLF the two sides GK and GC are respectively equal to the two HL and HF, and the

angle KGC to the angle LHF (b); therefore the bases are equal (c), KC to LF. And because the angle KAC is equal to LDF (b), the segment KBAC is similar to the segment LEDF (d); and they are upon equal straight lines KC and LF, therefore the segment KBAC is equal to the segment LEDF (e). Then taking the equal segments KBAC and LEDF from the equal circles ABC and DEF, the remaining segments and their arcs KC and LF are equal (f); and in like manner it may be shown that the arcs BK and EL are equal; therefore *the whole arc* BKC *is equal to the arc* ELF.

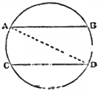


(b) Hypoth. and Constr.
(c) I. 4.
(d) III. Def. 10.
(e) III. 24.
(f) Ax. 3.

SCHOLIUM. In the enunciation of this proposition and some of the following, we have inserted the words "or the same circle," because that which obtains in equal circles must necessarily do so in the same circle. The demonstration as given above differs from Euclid's in the given angles being bisected. This is done to render the proof applicable when the given angles at the center are equal to or greater than two right angles, in which case Euclid's proof could not be applied.

COROLLARY 1. THEOREM. *If two chords* (AB and DC) *of a circle are parallel,* they intercept equal arcs.

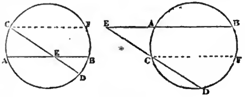
Join AD. Then because AD meets the parallels AB and CD, the alternate angles BAD and ADC are equal (a), and therefore the arcs AC and BD are equal (b).



(a) I. 29
(b) III. 26.

COROLLARY 2. THEOREM. *If two chords of a circle* (AB and CD) *meet one another,* the angle BED formed by them is equal to the angle (FCD) terminated at the circumference by the sum or difference of the arcs (AC and BD) which they intercept, according as the point in which they meet is within or without the circle.

Draw CF *parallel to* AB (a). Then when the point of intersection is within the circle, because the straight line CD meets the two parallels CF and AB, it forms the angle DEB equal to the internal and opposite angle DCF (b). But the arc FB is equal to AC (c), therefore FD is equal to the sum of the arcs AC and DB, and *the angle* BED *is*



(a) I. 31.
(b) I. 29.
(c) III. 26, cor. 1

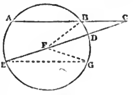
equal to the angle FCD, *terminated at the circumference by* FD, *the sum of the arcs.* Again, when the point of intersection is without the circle, because the straight line ED meets the two parallels EB and CF, it forms the angle DCF equal to the internal and opposite angle DEB (*b*). But the arc BF is equal to AC (*c*), therefore FD is equal to the difference of the arcs BD and AC, and *the angle* BED *is equal to the angle* FCD, *terminated at the circumference by* FD, *the difference of the arcs.*

COROLLARY 3. THEOREM. *If chords intersect at the same angle, within a circle,* the sums of the arcs which they respectively intercept are equal; *and if they intersect without the circle,* the differences are equal; *but if one pair intersect within, and the other without the circle,* the sum of the one pair of arcs is equal to the difference of the other.

COROLLARY 4. THEOREM. *If two chords intersect within a circle at right angles,* the sums of the opposite arcs intercepted are equal.

COROLLARY 5. THEOREM. *If a chord of a circle* (AB) *be produced till the produced part* (BC) *is equal to the radius, and if a line be drawn from its extremity through the center of the circle to meet the concave circumference,* the concave portion of the circumference intercepted (AE) is equal to three times the convex (BD).

Draw EG *parallel to* AB (*a*), *and join* FB *and* FG. Because in the triangle BFC the side BF is equal to BC, the angle BFC is equal to C (*b*); and because EC meets the two parallels AC and EG, the alternate angles E and C are equal (*c*); therefore the angle BFC is equal to E. But the angle CFG at the center is double of the angle E at the circumference (*d*); therefore the angle CFG is double of BFC. To each add BFC, and the angle BFG is equal to three times the angle BFC; therefore the arc BG is three times the arc BD. But the arc AE is equal to BG (*e*); therefore *the arc* AE *is three times the arc* BD.

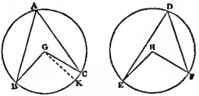


(*a*) I. 31.
(*b*) I. 5.
(*c*) I. 29.
(*d*) III. 20.
(*e*) III. 26, cor. 1.

PROPOSITION XXVII.

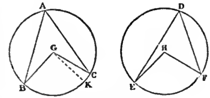
THEOREM.—*If angles* (BGC and EHF, or A and D) *stand upon equal parts of the circumferences of equal circles, or of the same circle,* they are equal to one another, whether they be at the center or the circumference.

For, if it be possible, let one of them BGC be the greater, and at G on the straight line BG form the angle BGK equal to EHF (*a*). Then because equal angles stand upon equal parts of the circumference, the arc BK is equal



(*a*) I. 23.

to EF (*b*); but BC is also equal to EF (*c*); therefore BK is equal to BC, the part equal to the whole, which is absurd; therefore *neither of the angles* BGC *and* EHF *is greater than the other, but they are equal; and the angles* A *and* D being respectively halves of the equals BGC and EHF, *are themselves equal.*



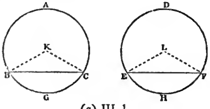
(*b*) III. 26.
(*c*) Hypoth.

SCHOLIUM. This proposition is the converse of the twenty-sixth.

PROPOSITION XXVIII.

THEOREM.—*If in equal circles, or the same circle, straight lines* (BC and EF) *are equal,* they cut off equal parts of the circumferences, the greater equal to the greater (BAC to EDF), and the less to the less (BGC to EHF).

CONSTRUCTION. *Find the centers* K *and* L *of the circles* ABC *and* DEF (*a*), *and join* KB, KC, LE, *and* LF.



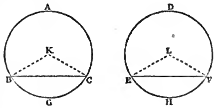
(*a*) III. 1.
(*b*) I. Def. 15.
(*c*) Hypoth.
(*d*) I. 8.
(*e*) III. 26.

DEMONSTRATION. Because the circles ABC and DEF are equal, their radii are equal (*b*); therefore in the triangles KBC and ELF the sides BK and KC are equal to EL and LF, and the base BC to EF (*c*), and the angle K is therefore equal to L (*d*). But equal angles stand upon equal parts of the circumference (*e*); therefore *the arc* BGC *is equal to* EHF; and because the whole circumference ABGC is equal to the whole circumference DEHF, therefore *the remaining arc* BAC *is equal to* EDF.

PROPOSITION XXIX.

THEOREM.—*If in equal circles, or the same circle, equal parts of the circumference* (BGC and EHF) *are taken*, they are subtended by equal straight lines.

CONSTRUCTION. *Find the centers* K *and* L *of the circles* ABC *and* DEF (*a*), *and join* KB, KC, LE, *and* LF.



(*a*) III. 1.
(*b*) III. 27.
(*c*) I. Def. 15.
(*d*) I. 4.

DEMONSTRATION. Because the arc BGC is equal to EHF, the angle K is equal to L (*b*); and because the circles ABC and DEF are equal, their radii BK, KC, and EL, LF, are equal (*c*); therefore in the triangles KBC and LEF *the base* BC *is equal to* EF (*d*).

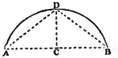
SCHOLIUM. This proposition is the converse of the twenty-eighth. The four preceding propositions may be united, and all expressed under the following more general form:—

THEOREM. *If in equal circles, or the same circle, either of the five pairs, namely, arcs, chords, angles at the center, angles at the circumference, or sectors, are equal*, the other four pairs are equal.

PROPOSITION XXX.

PROBLEM.—To bisect a given *arc* (ADB).

SOLUTION. *Join* AB, and bisect it in C (*a*); *from the point* C *draw* CD *perpendicular to* AB (*b*), *and join* AD, DB; *then the arc is bisected in* D.



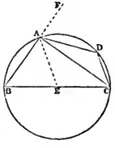
(*a*) I. 10.
(*b*) I. 11.
(*c*) Solution.
(*d*) Ax. 11.
(*e*) I. 4.
(*f*) III. 28.

DEMONSTRATION. Because in the triangles ACD and BCD the side AC is equal to BC (*c*), the side DC common to both, and the angle ACD equal to BCD (*d*), the base AD is equal to BD (*e*); but equal straight lines cut off equal arcs (*f*), therefore *the arc* AD *is equal to* DB, *and the arc* ADB *is bisected in* D.

PROPOSITION XXXI.

THEOREM.—*If in a circle* [1] *an angle* (BAC) *be in a semicircle,* it is a right angle; [2] *but if the angle* (ABC) *be in a segment greater than a semicircle,* it is less than a right angle; [3] *and if the angle* (ADC) *be in a segment less than a semicircle,* it is greater than a right angle

CONSTRUCTION. *Find the center* E *of the semicircle* BAC (*a*), *join* AE, *and produce* BA *to* F.



DEMONSTRATION. [1] Because in the triangle BAE the sides BE and AE are equal (*b*), the angles B and BAE are equal (*c*); and in the triangle EAC, because the sides EA and EC are equal (*b*), the angles EAC and ECA are equal (*c*); therefore the whole angle BAC is equal to the sum of the angles B and BCA. But the external angle FAC is also equal to the internal opposite angles B and BCA (*d*); therefore *the angle* FAC *is equal to* BAC, *and they are each equal to a right angle* (*e*).

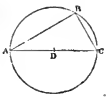
[2.] And because the two angles B and BAC of the triangle ABC are together less than two right angles (*f*), and BAC is a right angle, B must be less than a right angle; therefore *the angle in a segment* ABC *greater than a semicircle is less than a right angle.*

(*a*) III. 1.
(*b*) I. Def. 15.
(*c*) I. 5.
(*d*) I. 32 A.
(*e*) I. 13.
(*f*) I. 17.
(*g*) III. 22.

[3.] In the segment ADC take any point D, and join AD and DC. Then because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are equal to two right angles (*g*); therefore B and D are together equal to two right angles. But B is less than a right angle; therefore D is greater than a right angle; and therefore *the angle in the segment* ADC *less than a semicircle is greater than a right angle.*

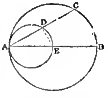
SCHOLIA. 1. This proposition might be easily proved from the twentieth; for since the angle at the center of a semicircle is equal to two right angles, the angle at the circumference is equal to one right angle. And in like manner, in a segment greater or *less* than a semicircle, as the angle at the center is less than or *greater* than two right angles, so is the angle at the circumference less than or *greater* than one right angle.

2. The converse of this proposition is true, namely:—*That if an angle in a segment is less than, equal to, or greater than a right angle*, the segment is greater than, equal to, or less than a semicircle. If the hypotenuse of a right-angled triangle ABC be bisected in D, and from D as a center a circle be described with the radius AD, it will pass through each of the angles of the triangle A, B, and C.



COROLLARY 1. THEOREM. *If a circle be described on the radius of another circle* (ABC), *and a straight line* (AC) *be drawn from the point* (A) *in which they meet to the outer circumference*, that line will be bisected by the interior one.

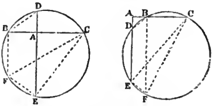
Join DE. Then because ADE is a semicircle, the angle ADE is a right angle (*a*); and because the straight line DE drawn from the center E is perpendicular to the chord AC, *it bisects it* (*b*).



(*a*) III. 31.
(*b*) III. 3 (2).

COROLLARY 2. THEOREM. *If from a point* (A), *within or without a circle, two lines be drawn at right angles to each other, to meet the circumference*, the sum of the squares on the segments (BA, CA, DA, and EA) between the point and the circumference is equal in area to the square on the diameter of the circle.

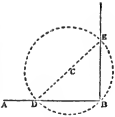
Draw BF *parallel to* DE (*a*), *and join* CE, CF, EF, *and* BD. Then because BF and DE are parallel, the intercepted arcs DB and EF are equal (*b*), and therefore the chords DB and EF are also equal (*c*). Then in the right-angled triangle BAD the sum of the squares on AB and AD is equal in area to the square on BD, or to the square on its equal EF (*d*). And in the right-angled triangle AEC the sum of the squares on AE and AC is equal in area to the square on CE. Then because AC meets the two parallels, it makes the angles EAC and FBC equal (*e*); therefore FBC is a right angle, and the segment FDC a semicircle (*f*); therefore the angle FEC is a right angle (*g*). Then in the right-angled triangle FEC the sum of the squares on EF and EC is equal in area to the square on the diameter FC (*d*); but the sum of the squares on EF and EC is equal in area to the sum of the squares on AB, AD, AC, and AE; therefore *the square on the diameter* FC *is equal in area to the sum of the squares on the segments* AB, AC, AD, *and* AE.



(*a*) I. 31.
(*b*) III 26, cor. 1
(*c*) III. 29.
(*d*) I. 47.
(*e*) I. 29.
(*f*) III. 31, schol. 2
(*g*) III. 31.

COROLLARY 3. PROBLEM. To draw a line through the extremity of a given *line* (AB), perpendicular to the same.

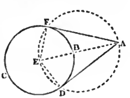
From any point C *without the given line, and with* CB *as a radius, describe a circle cutting* AB *in* D; *join* DC, *and produce the line to cut the circumference of the circle in* E; *draw* BE, *and it will be the perpendicular required.* For the angle DBE, being in the semicircle EBD, is a right angle (*a*).



(*a*) III. 31.

COROLLARY 4. PROBLEM. To draw a tangent to a given *circle* (DBC), from a given *point* (A) *without it.*

Find the center E *of the given circle* (*a*), *and join* AE; *upon* AE, *as a diameter, describe the circle* AFD, *cutting the circumference of the given circle in* F *and* D; *join* AD *and* AF; *then both* AD *and* AF *are tangents to the given circle.* For *join* FE *and* DE; then because the line AE bisects the circle AFD, both the angles F and D are in semicircles, and are therefore right angles (*b*). From the solution of this proposition it is evident two tangents can be drawn from any point without a circle to the same

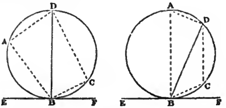


(*a*) III. 1.
(*b*) III. 31.

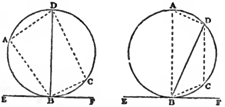
PROPOSITION XXXII.

THEOREM.—*If a straight line* (EF) *touches a circle, and from the point of contact* (B) *a straight line* (BD) *be drawn cutting the circle,* the angles (EBD and FBD) formed by this line and the line touching the circle are equal to the angles (BCD and BAD) in the alternate segments of the circle.

DEMONSTRATION. If the line BD is drawn at right angles to the line EF touching the circle, it passes through the center (*a*), and bisects the circle; therefore *the angles* A *and* C, being in the semicircles BAD and BCD, are right angles (*b*), and *are equal to the angles* EBD *and* FBD (*c*).



(*a*) III. 19.
(*b*) III. 31.
(*c*) Ax. 11.

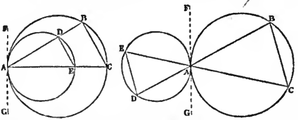


But if BD is not at right angles to EF, let AB be so drawn; then because ADCB is a semicircle, ADB is a right angle (*b*), and the other two angles A and DBA of the triangle DAB are together equal to a right angle (*d*), and therefore to the angle FBA, which is also a right angle. From these equals taking away the common angle DBA, *the remaining angles* A *and* FBD *are equal.* Again, in the quadrilateral figure ABCD, the opposite angles A and C are together equal to two right angles (*e*); but the angles EBD and FBD are also together equal to two right angles (*f*); therefore the angles A and C are equal to the angles EBD and FBD; and taking away the equals A and FBD, *the remaining angles* C *and* EBD *are equal.*

(*b*) III. 31.
(*d*) I. 32 B, cor. 2.
(*e*) III. 22.
(*f*) I. 13.

SCHOLIUM. The first case considered above, namely, that in which the line BD is drawn at right angles to the touching line, has been usually omitted in the Elements.

COROLLARY. THEOREM. *If two straight lines* (AB and AC) *be drawn through the point of contact* (A) *of two circles,* they intercept arcs, the chords of which (BC and DE) are parallel.



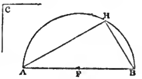
Through the point of contact A *draw the tangent* FG. When one circle is within the other, the angles ADE and ABC are each equal to the angle CAG (*a*); therefore they are equal to each other; then because the straight line AB meets the two straight lines DE and BC, and forms the external angle ADE equal to the internal opposite angle ABC, *the two lines* DE *and* BC *are parallel* (*b*). Again, when each circle is without the other, because the angles CAG and EAF are vertical, they are equal (*c*); but the angle CAG is equal to B (*a*), and the angle D to EAF (*a*); therefore the angles B and D are equal; and because the straight line DB, meeting the two straight lines ED and BC, forms the alternate angles B and D equal *the two lines* ED *and* BC *are parallel* (*d*).

(*a*) III. 32.
(*b*) I. 28 A.
(*c*) I. 15.
(*d*) I. 27.

PROPOSITION XXXIII.

Problem.—On a given *finite straight line* (AB) to describe a segment of a circle which shall contain an angle equal to a given *rectilineal angle* (C).

Let the given angle be a right angle:—

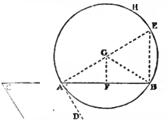


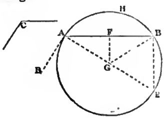
Solution. *Bisect* AB *in* F (*a*), *and from the center* F, *with the radius* AF, *describe the semicircle* AHB, *and it will be the segment required.*

(*a*) I. 10.
(*b*) III. 31.

Demonstration. For the angle H in the semicircle AHB is a right angle (*b*), and the given angle is a right angle; therefore *the angle in the segment* AHB *is equal to the given angle* C.

If the given angle be not a right angle:—





Solution. *At the point* A *in the straight line* AB *form the angle* BAD *equal to the given angle* C (*c*), *and from the point* A *draw* AE *perpendicular to* AD (*d*); *bisect* AB *in* F (*a*), *and from* F *draw* FG *perpendicular to* AB (*d*); *then from the center* G, *with the radius* GA, *describe the circle* ABEH, *and join* GB *and* BE. *Then the segment* AHB *shall be upon the given line* AB, *and shall contain an angle equal to the given angle* C.

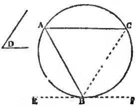
(*c*) I. 23.
(*d*) I. 11.
(*e*) I. 4.
(*f*) III. 16, cor
(*g*) III. 32.

Demonstration. Because in the triangles AFG and BFG the side AF is equal to BF, the side FG common to both, and the angle AFG equal to BFG, the base AG is equal to GB (*e*); therefore the circle AHE, described from G as a center, with the radius GA, shall pass through the point B; and because from the point A, at the extremity of the diameter AE, AD is drawn perpendicular to AE, therefore AD touches the circle (*f*); and because AB, drawn from the point of contact A, cuts the circle, the angle DAB is equal to the angle in the alternate segment AHB (*g*); but the

angle DAB is equal to the angle C; therefore *the angle in the segment* AHB *is equal to the angle* C, *and it is upon the straight line* AB

PROPOSITION XXXIV.

PROBLEM.—To cut off from a given *circle* (ABC) a segment which shall contain an angle equal to a given *rectilineal angle* (D).



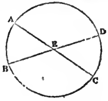
SOLUTION. *Draw the straight line* EF, *touching the circle* ABC *in the point* B (*a*), *and at the point* B *in the straight line* EF *form the angle* FBC *equal to the given angle* D (*b*); *then the segment* BAC *cut off of the given circle shall contain an angle equal to the given angle* D.

(*a*) III. 17.
(*b*) I. 23.
(*c*) III. 32.

DEMONSTRATION. Because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC is equal to the angle in the alternate segment BAC (*c*); but the angle FBC is equal to the angle D; therefore *the angle in the segment* BAC *cut off from the given circle is equal to the given angle* D.

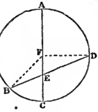
PROPOSITION XXXV.

THEOREM.—*If two straight lines* (AC and BD) *cut one another within a circle*, the rectangle under the segments (AE and EC) of one of them is equal in area to the rectangle under the segments (BE and ED) of the other.



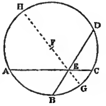
DEMONSTRATION. [1.] Let the two lines intersect in the center of the circle; then since AE, EC, BE, and ED are all equal (*a*), *the rectangle under* AE *and* EC *must be equal to the rectangle under* BE *and* ED.

(*a*) Def. 15



[2.] Let one of the given lines AC pass through the center, and the other not; *find the center* F *of the circle* (*b*), *and join* DF *and* BF. Then in the triangle FBD the rectangle under BE and ED is equal in area to the difference of the squares on BF and FE (*c*), that is, to the difference of the squares on CF and FE; then because AE is equal to the sum of the two lines CF and FE, and CE is equal to their difference, the rectangle under CE and AE is equal in area to the difference of the squares on CF and FE (*d*); therefore *the rectangle under* BE *and* ED *is equal in area to the rectangle under* CE *and* AE.

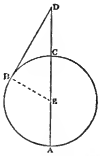
(*b*) III. 1.
(*c*) II. 6, cor. 2.
(*d*) II. 5, cor. 1.



[3.] Let neither of the given lines pass through the center; *find the center* F (*b*), *and through* E *and* F *draw the diameter* HG; then the rectangle under HE and EG has just been shown to be equal in area to the rectangle under AE and EC, and also equal in area to the rectangle under BE and ED; therefore *the rectangle under* AE *and* EC *is equal in area to the rectangle under* BE *and* ED.

SCHOLIUM. By a modification in the principle of demonstrating the second case, it is made to include the second and third cases of Simson, and thus reduces the whole number of cases from four to three, and greatly abridges the proof.

PROPOSITION XXXVI.



THEOREM.—*If from a point* (D) *without a circle two straight lines be drawn, one of which* (DA) *cuts the circle, and the other* (BD) *touches it*, the rectangle under the whole line which cuts the circle (DA) and the segment without the circle (DC), is equal in area to the square on the line (BD) which touches it.

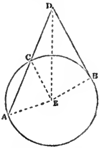
DEMONSTRATION. [1.] Let the line DA pass through the center E of the circle, *and join* EB; then in the right-angled triangle DBE the square on BD is equal in area to the difference of the squares on DE and BE (*a*), or to the

(*a*) I. 47.

difference of the squares on DE and CE; but because DA is equal to the sum of the two lines DE and CE, and DC is equal to their difference, the rectangle under DA and DC is equal in area to the difference of the squares on DE and CE (*b*); therefore *the square on* BD *is equal in area to the rectangle under* DA *and* DC.

(*b*) II. 5, cor. 1.

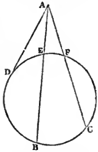
[2.] If the line DA does not pass through the center, *find the center* E (*c*), *and join* EA, EB, EC, and ED. Then in the isosceles triangle ACE the rectangle under DA and DC is equal in area to the difference of the squares on DE and CE (*d*), or to the difference of the squares on DE and EB; but in the right-angled triangle DEB the square on DB is equal in area to the difference of the squares on DE and EB; therefore *the square on* DB *is equal in area to the rectangle under* DA *and* DC.



(*c*) III. 1.
(*d*) II. 6, cor. 2.

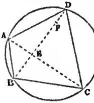
COROLLARY 1. THEOREM. *If from a point* (A) *without a circle two straight lines* (AB and AC) *be drawn cutting it,* the rectangles under the whole lines and the parts of them without the circle are equal in area to one another (AE and AB to AF and AC).

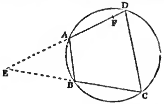
For each of them is equal in area to the straight line AD drawn from the same point to touch the circle (*a*), and therefore they are equal in area to one another



(*a*) III. 36.

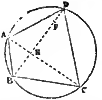
COROLLARY 2. THEOREM. *If the rectangles under the segments* (AE, EC, and BE, ED), *made by the intersection of the diagonals of a quadrilateral figure* (ABCD), *are equal in area, or if the rectangles under the segments* (EA, ED, and EB, EC), *made by producing its opposite sides to intersect, are equal in area,* the quadrilateral may have a circle described about it.

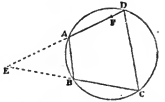




Describe a circle passing through three of the angles A, B, and C of the quadrilateral (*a*), then

(*a*) Theor. attached to III. 1.



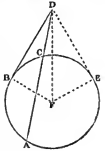


it shall also pass through the fourth angle D; for if not, let it cut ED in F, then the rectangle under AE and EC is equal in area to the rectangle under BE and EF (*b*); but the rectangle under AE and EC is also equal in area to the rectangle under BE and ED (*c*); therefore the rectangle under BE and EF is equal in area to the rectangle under BE and ED, the less equal to the greater, which is absurd; therefore *the circle passes through the four angles of the quadrilateral figure* ABCD.

(*b*) III. 35, or 36, cor.
(*c*) Hypoth.

PROPOSITION XXXVII.

Theorem.—*If from a point* (D) *without a circle two straight lines be drawn, one* (DA) *cutting the circle, and the other* (DB) *meeting it, and if the rectangle under the whole line which cuts the circle, and the part of it without the circle* (DA and DC), *be equal in area to the square on the line* (DB) *which meets it,* that line touches the circle.



Construction. *Draw the straight line* DE, *touching the circle* (*a*), *find the center* F *of the circle* (*b*), *and join* FB, FD, *and* FE.

Demonstration. Then the rectangle under DA and DC is equal in area to the square on DE (*c*); but the rectangle under DA and DC is also equal in area to the square on DB (*d*); therefore the square on DB is equal to the square on DE, and DB is equal to DE (*e*); then in the triangles FBD and FED, because the side DB is equal to DE, the side BF to FE, and the side DF common to both, the angle B is equal to the angle E (*f*); but the angle E is a right angle (*g*), therefore the angle B is a right angle *and the straight line* DB *meets the circle* (*h*).

(*a*) III. 17.
(*b*) III. 1.
(*c*) III. 36.
(*d*) Hypoth.
(*e*) I. 46, cor. 2.
(*f*) I. 8.
(*g*) Construct
(*h*) III. 16.

A CLASSIFIED INDEX

TO

THE FIRST THREE BOOKS

OF THE

ELEMENTS OF EUCLID.

THEOREMS.

A. *Of the Angles formed by the Meeting and Intersection of straight Lines.*

	HYPOTHESES.	CONSEQUENCES.
I. 11, cor.	If two lines be straight	They cannot have a common segment.
I. 13.	If a straight line standing upon another forms angles with it.	They are either two right angles, or are together equal to two right angles.
I. 14.	If two straight lines meet another straight line at the same point and on opposite sides, and make the adjacent angles with it together equal to two right angles.	Those two straight lines will form one continued straight line.
I. 16, cor. 2.	If two straight lines be drawn from any point to the same straight line.	They cannot both be perpendicular to it.
I. 16, cor. 1.	And if one be perpendicular to it, and the other not.	The one which is perpendicular shall be on the same side of the one which is not, as the acute angle.
I. 13, cor. 2.	If two straight lines intersect.	The four angles which they form at the point of intersection are together equal to four right angles.
I. 15.	Idem.	The vertical angles are equal.
I. 15, schol. 2.	If four straight lines meet in the same point, and make the vertical angles equal.	Each alternate pair of lines will form one continued straight line.
I. 13, cor. 1.	If several straight lines stand on the same side of another straight line at the same point, and make angles with it.	All those angles are together equal to two right angles.
I. 13, cor 3.	If any number of straight lines diverge from a point.	All the angles which they make, taken together, are equal to four right angles.

B. *Of Parallel straight Lines.*

	Hypotheses.	Consequences.
I. 27.	If a straight line intersect two other straight lines, both in the same plane, And form alternate angles equal to each other,	The two straight lines shall be parallel.
I. 28 A.	Or form an external angle equal to the internal and opposite angle upon the same side of the line,	
I. 28 B.	Or form internal angles at the same side equal to two right angles.	
I. 29.	If a straight line intersect two parallel straight lines.	It forms the alternate angles equal to one another, And the external angle equal to the internal and opposite angle upon the same side, And also the two internal angles on the same side, together equal to two right angles.
I. 29, theor.	If a straight line meets two straight lines so as to make the two internal angles on the same side together less than two right angles.	These straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.
I. 29, cor.	If two straight lines are parallel.	They are equidistant.
I. 30.	If two straight lines be parallel to the same straight line.	They are parallel to each other.
I. 33.	If two straight lines join the adjacent extremities of two equal and parallel straight lines.	They are themselves equal and parallel.

C. *Comparison of Triangles as to Equality.*

	Hypotheses.	Consequences.
I. 7.	If two triangles be upon the same base, and on the same side of it.	They cannot have their sides which are terminated in one extremity of that base equal to one another, and also those which are terminated in the other extremity.

	Hypotheses.	Consequences.
I. 32 B, cor. 3.	If two triangles have two angles in the one respectively equal to two angles in the other,	The remaining angles are also equal.
I. 26.	And a side of the one equal to a side of the other, either the sides adjacent to, or the sides opposite to, those equal angles.	The remaining angles and sides shall be respectively equal to one another, And the triangles themselves shall be equal to one another.
	If two triangles have two sides of the one respectively equal to two sides of the other,	
I. 8, and cor.	And havo also their bases equal;	The angles formed by the equal sides are equal, And the angles opposite the equal sides are equal, And the triangles themselves are eqnal.
I. 25.	But if the third side of the one be greater than the third side of the other.	The angle opposite to the greater side is greater than the angle which is opposite to the less.
	If two triangles have two sides of the one respectively eqnal to two sides of the other,	
I. 4.	And the angles formed by those sides also equal to one another;	Tfieir bases or third sides will be equal, And the angles at the bases, which are opposite to the equal sides, will be eqnal, And the triangles themselves will be equal.
I. 24.	But if the angle formed by two sides of one be greater than the angle formed by the two sides equal to them of the other.	The side opposite to that greater angle is greater than the side which is opposite to the less.
	If triangles are between the same parallels,	
I. 37.	And upon the same base,	
I. 38.	Or upon eqnal bases.	They are equal to one another in area.
	If triangles are eqnal in area,	
I. 39.	And upon the same base, and on the same side of it,	
I. 40.	Or upon equal bases in the same straight line, and on the same sido of it.	They are between the same parallels.

D. *On the Relations between the Sides and Angles of Triangles.*

	Hypotheses.	Consequences.
I. 20.	If two straight lines are the sldes of a triangle.	They are together greater than the third side.
I. 20, cor.	If the difference of any two sides of a triangle be taken.	It is less than the third side
I. 17.	If any two angles are those of a triangle.	They are together less than two right angles.

	Hypotheses.	Consequences.
I. 32 B.	If any three angles are the internal angles of a triangle.	They are together equal to two right angles.
I. 32 B, cor. 2.	If one angle of a triangle be equal to the other two.	It is a right angle.
I. 32 B, cor. 1.	If one angle of a triangle be a right angle.	Neither of the other angles can be a right angle.
I. 32 B, cor. 2.	Idem.	The other two are together equal to a right angle.
I. 6.	If two angles of a triangle are equal.	The sides opposite to those angles are also equal.
I. 19.	If in any triangle one angle is greater than another.	The side which is opposite to the greater angle is greater than the side which is opposite to the less.
I. 5.	If a triangle be isosceles.	The angles at the base are equal to one another.
	And if the equal sides be produced.	The angles formed by the produced sides and the base, below the same, shall be equal.
I. 32 B, cor. 4.	If a right-angled triangle be isosceles.	Each angle at the base is half a right angle.
I. 18.	If one side of any triangle be greater than another.	The angle opposite to the greater side is greater than the angle which is opposite to the less.
I, 5, cor.	If a triangle is equilateral.	It is equiangular.
I. 32 B, cor. 5.	Idem.	Each angle is equal to two-thirds of a right angle.
I. 6, cor.	If a triangle is equiangular.	It is equilateral.
I. 16.	If one side of a triangle be produced.	The external angle is greater than either of the internal opposite angles.
I. 32 A.	Idem.	The external angle is equal to the sum of the two internal and opposite angles.
I. 32 B, schol. 3.	If an exterior angle of a triangle be bisected, and also one of the interior and opposite angles.	The angle formed by the bisecting lines is equal to half the other interior and opposite angle of the triangle.
I. 47.	If a triangle be right-angled.	The square which is constructed upon the side subtending the right angle is equal in area to the sum of the squares constructed upon the sides which form the right angle.
I. 47, cor. 8.	Idem.	Any figure which is constructed upon the side subtending the right angle is equal in area to the sum of the similar figures constructed upon the sides which form the right angle.

	HYPOTHESES.	CONSEQUENCES.
I. 47 schol. 3.	If parallelograms be constructed upon two of the sides of any triangle, and their sides parallel to the sides of the triangle be produced to meet in a point; if a straight line be drawn from that point to the vertex of the triangle, and if a parallelogram be constructed upon the base of the triangle whose other sides are equal and parallel to that straight line.	The last parallelogram is equal in area to the two former.
I. 48.	If the square constructed upon one side of a triangle be equal in area to the sum of the squares constructed upon the other two sides.	The angle opposite to that side is a right angle.
II. 4, cor. 3.	If from either end of the hypotenuse of a right-angled triangle parts be cut off equal to the adjacent sides.	The square on the middle segment thus formed is equal in area to twice the rectangle under the extreme segment.

E. *On the Relations of Lines drawn in Triangles.*

	HYPOTHESES.	CONSEQUENCES.
I. 38, cor.	If a straight line from the vertex of a triangle bisects its base.	It also bisects the triangle.
II. 9, cor. 1.	Idem.	The sum of the squares on the two sides is equal in area to double the sum of the squares on half the base and on the bisecting line.
I. 26, cor. 2.	If a straight line from the vertex of an isosceles triangle bisects its base.	It is perpendicular to it, And it bisects the opposite angle.
	If a straight line bisects the angle opposite to the base of an isosceles triangle.	It bisects the base, And it is perpendicular to it.
II. 6, cor. 2.	If a straight line be drawn from the vertex of an isosceles triangle to any point in the base or the base produced.	The rectangle under the segments of the base is equal in area to the difference between the square on this line and the square on either side of the triangle.

	Hypotheses.	Consequences.
I. 47, cor. 1.	If in any triangle a perpendicular to the base be drawn from the vertex.	The difference of the squares on the sides is equal in area to the difference of the squares on the segments of the base.
I. 47, cor. 2.	Idem.	The sum of the squares on one side and the alternate segment is equal in area to the sum of the squares on the other side and the alternate segment.
II. 5, cor. 2.	Idem.	The difference of the squares on the sides is equal in area to twice the rectangle under the base and the distance of its middle point from the perpendicular.
I. 26, cor. 1.	If in an isosceles triangle a perpendicular to the base be drawn from the vertex.	It will bisect the base, And also the angle opposite to the base.
II. 13, cor. 1.	If in an isosceles triangle a perpendicular be drawn from either angle of the base to the opposite side.	Double the rectangle under that side and the segment between the perpendicular and the base is equal in area to the square on the base.
II. 12.	If a perpendicular be drawn from any of the acute angles of an obtuse-angled triangle to the opposite side produced.	The square on the side subtending the obtuse angle is greater than the sum of the squares on the two sides which contain the obtuse angle, by double the rectangle under the side, which is produced, and the external segment between the obtuse angle and the perpendicular.
II. 12. cor.	If in any obtuse-angled triangle the sides which contain the obtuse angle be produced, and perpendiculars be drawn to the acute angles.	The rectangle under one of those sides and the produced part between the obtuse angle and the perpendicular, is equal in area to the rectangle under the other side and its produced part.
II. 13.	If in any triangle a perpendicular be drawn to one of the sides which contains an acute angle, from the opposite angle.	The square on the side subtending that acute angle is less than the sum of the squares on the sides which contain that angle, by double the rectangle under the side to which the perpendicular is drawn, and the segment between the perpendicular and the acute angle.

	HYPOTHESES.	CONSEQUENCES.
I. 40, cor. 2. .	If a straight line bisects the two sides of a triangle.	It is parallel to the base.
I. 21.	If from a point within a triangle two straight lines be drawn to the extremities of any side.	They are together less than the sum of the two other sides of the triangle, And they form a greater angle.
I. 40, cor. 3. .	If three straight lines be drawn through the points of bisection of the three sides of a triangle.	They divide it into four equal triangles.
I. 40, cor. 4. .	Idem.	The three lines are respectively equal to half the parallel sides.

F. *Comparison of Parallelograms with Triangles.*

	HYPOTHESES.	CONSEQUENCES.
	If a parallelogram and a triangle be between the same parallels,	The parallelogram is double of the triangle.
I. 41.	And upon the same base,	
I. 41, cor. . .	Or upon equal bases.	

G. *Comparison of Parallelograms as to Equality.*

	HYPOTHESES.	CONSEQUENCES.
I. 34, cor. 2. .	If two parallelograms have an angle of the one equal to an angle of the other.	The remaining angles shall be respectively equal.
	If parallelograms are between the same parallels,	They are equal in area.
I. 35.	And upon the same base,	
I. 36.	Or upon equal bases.	
I. 46, cor. 1. .	If two squares are constructed on equal straight lines.	They are equal to one another.
I. 46, cor. 2. .	If two squares are equal. . .	Their sides are equal.

H. *On the Relations between the Sides, Angles, and Surfaces of Parallelograms.*

	HYPOTHESES.	CONSEQUENCES.
I. 34, theor. . .	If in any four-sided figure the opposite sides are equal, Or the opposite angles are equal.	It is a parallelogram

	Hypotheses.	Consequences.
I. 34.	If a figure be a parallelogram.	The opposite sides are equal to one another, The opposite angles are equal to one another, And the parallelogram is bisected by its diagonal.
I. 34, cor. 3.	Idem.	Its diagonals bisect each other.
I. 34, cor. 1.	If a parallelogram have one right angle.	All its angles are right angles.
I. 43.	If about the diagonal of a parallelogram two other parallelograms are formed.	Their complements are equal in area.
I. 43, cor.	Idem.	Those parallelograms and their complements are equiangular with the original parallelogram.
II. 4, cor. 4.	If parallelograms are about the diagonal of a square.	They are themselves squares.
I. 46, theor.	If a four-sided figure be a square.	All its angles are right angles.
I. 40, cor. 5.	If the sides of a four-sided figure be bisected, and the points of bisection of each pair of conterminous sides joined by straight lines.	Those lines will form a parallelogram whose area is equal to half that of the four-sided figure.
II. 9, cor. 3.	If straight lines be drawn from any point to the opposite angles of a rectangle.	The sum of the squares on the lines to one pair of opposite angles is equal in area to the sum of the squares on the lines to the other pair of opposite angles.
II. 9, cor. 4.	If a four-sided figure be taken.	The sum of the squares on the sides is equal in area to the sum of the squares on the diagonals, together with four times the square on the line joining their middle points.
II. 9, cor. 5.	If a parallelogram be taken.	The squares on the sides are together equal in area to the sum of the squares on the diagonals.

I. *Comparison of Rectangles contained by straight Lines and their Segments.*

	Hypotheses.	Consequences.
II. 2.	If a straight line be divided into any two parts.	The rectangles under the whole line, and each of the parts, are together equal in area to the square on the whole line.

	Hypotheses.	Consequences.
II. 3.	If a straight line be divided into any two parts.	The rectangle under the whole line, and one of those parts, is equal in area to the square on that part together with the rectanglo under the two parts.
II. 4.	If a straight line be divided into any two parts.	The sqnare on the whole line is equal in area to the sum of the squares on the parts, together with twice the rectangle under the parts.
II. 7.	Idem.	The sum of the square on the whole line and the square on either segment is equal in area to double the rectangle under the whole line and that segment, together with the square on the other segment.
II. 8.	Idem.	The square on the sum of the whole line and either segment is equal in area to four times the rectanglo under the whole line and that segment, together with the square on the other segment.
II. 4, cor. 1.	If a straight line be divided into any number of segments.	The sqnare on the whole line is equal in area to the sum of the sqnares upon the segments, together with twice the rectangle under each pair of segments.
II. 5.	If a straight line be bisected, and also cut into two nnequal parts.	The rectangle under the nnequal parts, together with the square on the line between the points of section, is equal in area to the square on half the line.
II. 9.	Idem.	The sum of the sqnares on the uneqnal parts is equal in area to double the sum of the square on half the line and the sqnare on tho line between the points of section.
II. 6.	If a straight line be bisected, and also produced to any point.	The rectangle under the whole line thus prodnced and the prodnced part, together with the square on half the line bisected, is eqnal in area to the square on the straight line which is made np of the half and the prodnced part

	Hypotheses.	Consequences.
II. 10.	If a straight line be bisected, and also produced to any point.	The square on the whole line thus produced, together with the square on the produced part, is equal in area to double the square on half the line bisected, together with double the square on the straight line, which is made up of the half and the produced part.
II. 11, cor . .	If a line be cut in extreme and mean ratio.	The greater segment will be cut in the same manner by taking on it a part equal to the less.
II. 4, cor. 2. . .	If a straight line be divided into three parts.	The squares on the sums of each of the extreme segments taken with the middle segment, together with twice the rectangle under the extreme segment, are equal in area to the squares on the whole line and the middle segment.
II. 1.	If there be two straight lines, one of which is divided into any number of parts.	The rectangle under the two lines is equal in area to the sum of the rectangles under the undivided line and the several parts of the divided line.
II. 1, cor. . .	If two straight lines be each of them divided into any number of parts.	The rectangle under the two lines is equal in area to the sum of all the rectangles under all the parts of the one, taken separately with all the parts of the other.
II. 5, schol. 2. .	If two straight lines be taken.	The square on half their sum is equal in area to the rectangle under them, together with the square on half their difference.
II. 5, cor. 1. .	Idem.	The rectangle under their sum and difference is equal in area to the difference of the squares on the two lines.
II. 7, cor. .	Idem.	The sum of the squares upon each of them is equal in area to twice the rectangle under them, together with the square on their difference.
II. 10, cor. .	Idem.	The sum of the squares upon each of them is equal in area to twice the square on half their sum, together with twice the square on half their difference.

	HYPOTHESES.	CONSEQUENCES.
II. 6, cor. 1. . .	If three straight lines are in arithmetical proportion.	The rectangle under the extremes, together with the square on the common difference, is eqnal in area to the square on the mean.

K. *Of Polygons.*

	HYPOTHESES.	CONSEQUENCES.
I. 32 B, cor. 7. .	If a figure be rectilinear. . .	The sum of all the internal angles, together with four right angles, is equal to twice as many right angles as the figure has sides.
I. 32 B, cor. 8. .	Idem.	All its external angles are together equal to four right angles.

L. *Relative to Circles generally.*

	HYPOTHESES.	CONSEQUENCES.
III. 2.	If any two points be taken in the circnmference of a circle.	The straight line which joins them falls within the circle.
III. 1, theor. .	If three points are not in the same straight line.	A circle may be described whose circumference shall pass through them.
III. Lemma. .	If a straight line bisects the chord of a circle perpendicnlarly.	It contains the center of that circle.
III. 29, schol. .	If in equal circles, or the same circle, either of the five pairs, namely, arcs, chords, angles at the center, angles at the circnmference, or sectors, are equal.	The other four pairs are equal.

M. *On the Relations of straight Lines drawn from any Point to the Circumference of a Circle.*

	Hypotheses.	Consequences.
III. 7.	If from any point within a circle, which is not the center, straight lines be drawn to the circumference.	The greatest is that which passes through the center. The remaining part of the diameter is the least. That line which is nearer to the line passing through the center is greater than one more remote. And more than two straight lines cannot be drawn which shall be equal.
III. 7, schol.	Idem.	Those lines which form equal angles with the line passing through the center are equal.
III. 9.	If a point be taken within a circle, from which more than two equal straight lines can be drawn to the circumference.	That point is the center of the circle.
III. 8.	If from any point without a circle straight lines be drawn to the circumference.	Of those which fall on the concave circumference, the greatest is that which passes through the center. Of the rest, that which is nearer to the line passing through the center is greater than the more remote. But of those which fall on the convex circumference, the least is that which, if produced, would pass through the center. Of the rest, that which is nearer to the least is less than the more remote. And more than two straight lines cannot be drawn, either to the concave or convex circumference, which shall be equal.
III. 8, schol.	Idem.	Those lines which form equal angles with the line passing through the center are equal.
III. 36.	If from a point within a circle two straight lines be drawn, one of which cuts the circle, and the other touches it.	The rectangle under the whole line which cuts the circle and the segment without the circle, is equal in area to the square on the line which touches it.

	HYPOTHESES.	CONSEQUENCES.
III. 36, cor. 1.	If from a point without a circle two straight lines be drawn, cutting it.	The rectangles under the whole lines, and the parts of them without tho circle, are equal in area to ono another.
III. 37	If from a point without a circle two straight lines be drawn, one cutting the circle and the other meeting it, and if the rectangle under the whole line which cuts the circle, and the part of it without the circle, be equal in area to the square on the line which meets it.	That line is a tangent to the circle.
III. 31, cor. 2	If from a point within or without a circle two straight lines be drawn at right angles to each other, to meet the circumference.	The sum of the squares on the segments between the point and the circumference is equal in area to the square on the diameter ot the circle.
III. 3, cor.	If from any point in the diameter of a circle or its extensions straight lines be drawn to the end of a parallel chord.	The squares on those lines are together equal in area to the sqnares on the segments into which the point divides the diameter.
III. 4.	If in a circle two straight lines cut one another, which do not both pass through the center.	They do not bisect each other.
III. 35.	If two straight lines cut one another within a circle.	The rectangle under the segments of ono of them is equal in area to the rectangle under the segments of the other.
III. 3	If a straight line drawn through the center of a circle bisect a straight line which does not pass through the center,	It is perpendicular to it.
	And if it is perpendicular to it.	It bisects it.
II. 9, cor. 2.	If from the middle point of a finite straight line as a center a circle be described, and lines be drawn from any point in its circumference to the extremities of the line.	The sum of the squares on those lines is always the same, and equal in area to double the snm of the squares on the radius and half the given line.
II. 14, schol.	If a perpendioular be drawn from any point in the circumterence of a semicircle to the diameter.	The square on the perpendicular is equal in area to the rectangle under the segments into which it divides the diameter.

N. *On the mutual Contact of a straight Line and a Circle.*

	Hypotheses.	Consequences.
III. 2, cor. 1.	If a straight line cut the circumference of a circle.	It cannot do so in more than two points.
III. 2, cor. 2.	If a straight line touches the circumference of a circle.	It meets it in only one point.
III. 18.	Idem.	The straight line drawn from the center to the point of contact shall be perpendicular to the line touching the circle.
III. 16, cor. 1.	If tangents are drawn from the same point to a circle.	They are equal to one another.
III. 16, cor. 2.	If tangents are at the extremities of the same diameter.	They are parallel to one another.
III. 19.	If a straight line touches the circumference of a circle, and a straight line be drawn perpendicular to it from the point of contact.	The center of the circle shall be in that line.
III. 16.	If a straight line be drawn from the extremity of the diameter of a circle, perpendicular to the same,	It will fall without the circle.
	And if any straight line be drawn from a point between that perpendicular and the circle, to the point of contact.	It will cut the circumference of the circle.
III. 31, cor. 1.	If a circle be described on the radius of another circle, and a straight line be drawn from the point in which they meet to the outer circumference.	That line will be bisected by the interior one.

O. *On the mutual Contact of two Circles.*

	Hypotheses.	Consequences.
III. 10.	If two lines be the circumferences of two circles.	They cannot cut one another in more than two points.
III. 5.	If the circumferences of two circles cut one another.	They have not the same center.
III. 6.	If the circumference of one circle touch the circumference of another circle internally in any point.	They have not the same center.
III. 11.	Idem.	The straight line joining their centers, being produced, shall pass through that point.

	Hypotheses.	Consequences.
III. 12. . . .	If the circumference of two circles touch each other externally in any point.	The straight line joining their centers shall pass through that point.
	If the circumference of one circle touch the circumference of another circle, either internally or externally.	There can only be one point of contact.
III. 19, cor. 1. .	Idem.	They have the same tangent at the point of contact.

P. *On the Angles in a Circle.*

	Hypotheses.	Consequences.
III. 26. . . .	If equal angles are in equal circles, or in the same circle.	They shall stand upon equal parts of the circumference, whether they be at the center or the circumference.
III. 27. . . .	If angles stand upon equal parts of the circumferences of equal circles, or of the same circle.	They are equal to one another, whether they be at the center or the circumference.
III. 21. . . .	If angles are in the same segment of a circle.	They are equal to one another.
III. 31. . . .	If, in a circle, an angle be in a semicircle.	It is a right angle.
	But if the angle be in a segment greater than a semicircle.	It is less than a right angle.
	And if the angle be in a segment less than a semicircle.	It is greater than a right angle.
III. 20. . . .	If an angle at the center of a circle have the same part of the circumference for its base as an angle at the circumference.	The former angle is double the latter.
III. 32. . . .	If a straight line touches a circle, and from the point of contact a straight line be drawn cutting the circle.	The angles formed by this line and the line touching the circle are equal to the angles in the alternate segments of the circle.

Q. *On Segments and their Chords.*

	Hypotheses.	Consequences.
III. 23. . . .	If two segments of circles are upon the same straight line, and upon the same side of it.	They cannot be similar without coinciding with one another.

	Hypotheses.	Consequences.
III. 24.	If two segments of circles are similar, and upon equal straight lines.	They are equal to one another, and have equal arcs.
III. 14.	If two straight lines in a circle are equal,	They are equally distant from the center.
	And if straight lines are equally distant from the center.	They are equal to one another.
III. 15.	If straight lines be drawn in a circle, of which one passes through its center.	That line is the greatest, And of all others, that which is nearer to the center is greater than the more remote, And the greater is nearer to the center than the less.
III. 26, cor. 1.	If two chords of a circle are parallel.	They intercept equal arcs.
III. 26, cor. 2.	If two chords of a circle meet one another.	The angle formed by them is equal to the angle terminated at the circumference by the sum or difference of the arcs which they intercept, according as the point in which they meet is within or without the circle.
III. 26, cor. 3.	If chords intersect at the same angle, within a circle,	The sums of the arcs which they respectively intercept are equal;
	And if they intersect without the circle,	The differences are equal;
	But if one pair intersect within, and the other without the circle.	The sum of the one pair of arcs is equal to the difference of the other.
III. 26, cor. 4.	If two chords intersect within a circle at right angles.	The sums of the opposite arcs intersected are equal.
III. 26, cor. 5.	If a chord of a circle be produced till the produced part is equal to the radius, and if a line be drawn from its extremity through the center of the circle to meet the concave circumference.	The concave portion of the circumference intercepted is equal to three times the convex.
III. 28.	If, in equal circles, or the same circle, straight lines are equal.	They cut off equal parts of the circumferences, the greater equal to the greater, and the less to the less.
III. 29.	If in equal circles, or the same circle, equal parts of the circumference are taken.	They are subtended by equal straight lines.
III. 32, cor.	If two straight lines be drawn through the point of contact of two circles.	They intercept arcs the chords of which are parallel.

R. *On Figures contained in Circles.*

	HYPOTHESES.	CONSEQUENCES.
III. 22.	If a four-sided figure is contained within a circle.	Its opposite angles are together equal to two right angles.
III. 22, cor. 1.	If one side of a four-sided figure contained within a circle be produced.	The external angle is equal to the angle opposite to the internal adjacent anglo.
III. 22, schol. 2.	If a four-sided figure has its opposito angles together equal to two right angles.	A circle may be described about it.
III. 36, cor. 2.	If the rectangle under the segments, made by the intersection of the diagonals of a four-sided figure, are equal in area, Or if the rectangles under the segments, mado by producing its opposite sides to intersect, are equal in area.	The four-sided figure may have a circle described about it.

PROBLEMS.

A. *Relating to straight Lines.*

I. 2.	From a given *point* to draw a straight line equal to a given *finite straight line.*
I. 31.	Through a given *point* to draw a straight line parallel to a given *straight line.*
I. 3.	From the greater of *two* given *straight lines* to cut off a part equal to the less.
I. 10.	To bisect a given *finite straight line.*
II. 11.	To divide a given *finite straight line* into two parts, so that the rectangle under the whole line and one segment shall be equal in area to the square on the other segment.
I. 34. schol.	To divide a given *finite straight line* into any given *number* of equal parts.

B. *Relating to rectilineal Angles.*

I. 23.	At a given *point* in a given *straight line* to form a rectilineal angle equal to a given *rectilineal angle.*
I. 9.	To bisect a given *rectilineal angle.*
I. 11.	From a given *point* in a given *straight line* to draw a perpendicular to that line.
I. 12.	To draw a straight line perpendicular to a given *straight line of an unlimited length,* from a given *point without it.*
III. 31, cor. 3.	To draw a straight line through the extremity of a given *straight line,* perpendicular to the same.
I. 32 B, cor. 6.	To bisect a given *right angle.*

C. *Relating to Triangles.*

D. *Relating to Parallelograms.*

E. *Relating to Circles.*